

Four-loop free energy for the 2D $O(n)$ nonlinear σ -model with 0-loop and 1-loop Symanzik improved actions*

B. Allés, M. Pepe

*Dipartimento di Fisica, Università di Milano-Bicocca
and INFN Sezione di Milano, Milano, Italy*

Abstract

We calculate up to four loops the free energy of the two-dimensional (2D) $O(n)$ nonlinear σ -model regularized on the lattice with the 0-loop and 1-loop Symanzik improved actions. An effective coupling constant based on this calculation is defined.

05.50.+q; 11.15.Ha; 12.38.Bx; 75.10.Hk; 11.10.Kk

I. INTRODUCTION

Numerical simulations on the lattice are a convenient method to compute expectation values of dimensionful operators in a field theory. To convert the expectation value obtained from the simulation into a matrix element written in physical units we must fix the lattice spacing a . In the present paper we will focus our attention on the case of asymptotically free field theories.

The lattice scale a is usually fixed through the integration of the beta function of the theory. This function can be evaluated in perturbation theory in which case a is written also as a weak coupling expansion (starting from three loops). It is said that the asymptotic scaling regime has been attained if the first universal terms in the perturbative expansion of the beta function are enough to have a good knowledge of the lattice spacing a . In principle we may expect that the better the weak expansion of the beta function is known, the more accurate the measurement of a dimensionful physical observable will be. However this hope usually fails due to the bad convergence properties of the weak expansion. In particular, the asymptotic scaling regime is barely achieved.

A cure to this problem may come from a redefinition of the expansion parameter in the perturbative series [1–3]. Usually the expansion parameter is the bare coupling constant g . However we may define any other parameter g_E related to the old one by an expansion $g_E = g + O(g^2)$. If g_E can be numerically determined by some non-perturbative procedure then the beta function, expressed as a power series in terms of g_E , can be regarded as a resummation where the non-perturbative effects have been absorbed in the very definition of the new parameter g_E . The problem is the choice of such a non-perturbative procedure that would produce a good resummation and a rapid convergence of the power series in g_E . The calculation scheme where g is substituted by g_E shall hereafter be called “effective scheme”.

In Ref. [4] the 2D $O(n)$ nonlinear σ -model has been analysed. This is an asymptotically free field theory, which shares several physical properties with four-dimensional (4D) Yang–Mills theory, among others the spontaneous generation of a mass in addition to the asymptotic freedom itself. In [4] the authors argued that the density of action (the internal energy) is a good operator to define g_E (actually the choice of this operator was proposed in the seminal papers [1,2] for 4D gauge theories as well as for the 2D nonlinear σ -model). In particular the corrections to asymptotic scaling in this effective scheme vanish in the large n limit. In Ref. [5], also devoted to the study of the 2D nonlinear σ -model, two different local operators were used to define the new expansion parameter and in both cases a clear improvement was observed. Moreover the results obtained for the measured quantities in the two effective schemes were compatible with each other, which means that the implicit resummations of the series were equivalent. In the case of gauge theories other types of redefinitions of the expansion parameter can be used, see for instance [6].

Nevertheless in many cases the lack of asymptotic scaling does not suffice to explain the poor results obtained from a Monte Carlo simulation. The lattice regularization of any operator (for example the action) differs from its continuum counterpart by terms of higher order in the lattice spacing a . If these terms are sizeable then the scaling ratios of operator matrix elements as evaluated on the lattice do not behave as prescribed by the continuum Renormalization Group equations, i.e.: they do not show physical scaling. This is another source of systematic errors in the Monte

*Preprint Bicocca-FT-99-04.

Carlo determination of any dimensionful observable. This problem can be mitigated if the Monte Carlo simulation is performed with a Symanzik improved action.

This difficulty occurs again in the 2D nonlinear σ -model. For instance, the calculation of the mass gap in this model with $O(3)$ symmetry has yielded results [7–15] in conflict with the exact analytical value m_{HMN} of Ref. [16]. In Ref. [17] an energy-based effective scheme was used to extract the mass gap but the deviation between the Monte Carlo and exact results was still beyond 10%. A definite improvement was obtained only after the problem concerning the lack of physical scaling was also tackled. In Ref. [5] a combination of an effective scheme together with a Symanzik improved action allowed an agreement between the Monte Carlo and the exact results well within 2–3%.

Therefore one of the main conclusions of Ref. [5] was that the combination of an effective scheme together with a Symanzik improved action may allow to obtain results reasonably clean from all lattice artifacts (assuming that the lattice volume is large enough). Our purpose is to further check this conclusion. To this end we want to calculate the mass gap of the 2D $O(3)$ nonlinear σ -model on the lattice by simulating two improved actions, the tree-order Symanzik and the 1-loop Symanzik actions [18,19]. Let us call Δm the difference between the mass extracted from a Monte Carlo simulation and the exact result of Ref. [16]. We want to study the dependence of Δm on the level of improvement in the action as well as on the number of corrective terms to the asymptotic scaling in the perturbative expansion of the lattice spacing. Firstly we need to calculate these corrections in perturbation theory and this is done in Ref. [20] where we calculate the lattice beta function β^L for the two above-mentioned Symanzik actions up to four loops in terms of the bare coupling g . In the present paper we will show the result for β^L in terms of an effective coupling g_E defined through the internal energy E . We will perform the calculation for a generic symmetry group $O(n)$, the specialization to the case $n = 3$ being trivial.

Our interest is actually twofold because a high precision determination of the mass in the model would also contribute to settle a long-standing debate about the validity of the m_{HMN} value [21].

In the following section we give the expressions of the two Symanzik improved actions and the explicit form of the operators for E . We also briefly describe our calculation procedure. In the third section we introduce some notation and a list of useful identities that we have used. The final results for the internal energy E will be given in section 4 together with the calculation of the lattice beta function in terms of g_E . A summary of the results and a brief account of the checks performed on them is given in the conclusions. Several appendices are devoted to the finite integrals that appear in the final expressions; we list these integrals and their numerical values, explaining how they were evaluated and showing some identities which relate them.

II. THE CALCULATION

In this section we give an outline of the calculation method that we have followed. The action of the nonlinear $O(n)$ -symmetric σ -model on the lattice can be written in many different ways. All of them share the same naïve continuum limit, i.e. the action of the model in the continuum

$$S^{\text{continuum}} = \frac{1}{2g} \int d^2x \sum_{\mu} \left(\partial_{\mu} \vec{\phi}(x) \right)^2, \quad (2.1)$$

where g is the coupling constant and $\vec{\phi}$ is a scalar field with n components constrained by the condition

$$\left(\vec{\phi}(x) \right)^2 = 1 \quad \text{for all } x. \quad (2.2)$$

In this work we shall consider three actions on the lattice. The standard action is the simplest one but yields the poorest results in simulations; on the other hand the Symanzik improved actions allow a progressive elimination of lattice artifacts as powers of the lattice spacing a and its logarithm $\log a$ [18,19]. These actions are

$$\begin{aligned} S^{\text{standard}} &= \frac{a^2}{g} \sum_x \frac{1}{2} \vec{\phi}(x) \cdot K_1 \cdot \vec{\phi}(x), \\ S^{0\text{-Symanzik}} &= \frac{a^2}{g} \sum_x \left(\frac{2}{3} \vec{\phi}(x) \cdot K_1 \cdot \vec{\phi}(x) - \frac{1}{24} \vec{\phi}(x) \cdot K_2 \cdot \vec{\phi}(x) \right), \\ S^{1\text{-Symanzik}} &= \frac{a^2}{g} \sum_x \left[\frac{1}{2} \vec{\phi}(x) \cdot K_1 \cdot \vec{\phi}(x) - a^2 c_5 g \left(K_1 \cdot \vec{\phi}(x) \right)^2 - a^2 \left(c_6 g - \frac{1}{24} \right) \sum_{\mu} \left(\partial_{\mu}^+ \partial_{\mu}^- \vec{\phi}(x) \right)^2 \right] \end{aligned}$$

$$\begin{aligned}
& -a^2 c_7 g \left(\vec{\phi}(x) \cdot K_1 \cdot \vec{\phi}(x) \right)^2 - a^2 c_8 g \sum_{\mu} \left(\vec{\phi}(x) \cdot \partial_{\mu}^+ \partial_{\mu}^- \vec{\phi}(x) \right)^2 \\
& - \frac{1}{16} a^2 c_9 g \sum_{\mu\nu} \left((\partial_{\mu}^+ + \partial_{\mu}^-) \vec{\phi}(x) \cdot (\partial_{\nu}^+ + \partial_{\nu}^-) \vec{\phi}(x) \right)^2 \Big], \tag{2.3}
\end{aligned}$$

where the superscript i -Symanzik denotes the i -loop Symanzik improved action. The several operators in this equation are defined by

$$\begin{aligned}
K_1 \cdot \vec{\phi}(x) &\equiv \frac{1}{a^2} \sum_{\mu} \left(2\vec{\phi}(x) - \vec{\phi}(x + \hat{\mu}) - \vec{\phi}(x - \hat{\mu}) \right), \\
K_2 \cdot \vec{\phi}(x) &\equiv \frac{1}{a^2} \sum_{\mu} \left(2\vec{\phi}(x) - \vec{\phi}(x + 2\hat{\mu}) - \vec{\phi}(x - 2\hat{\mu}) \right), \\
\partial_{\mu}^+ \vec{\phi}(x) &\equiv \frac{1}{a} \left(\vec{\phi}(x + \hat{\mu}) - \vec{\phi}(x) \right), \\
\partial_{\mu}^- \vec{\phi}(x) &\equiv \frac{1}{a} \left(\vec{\phi}(x) - \vec{\phi}(x - \hat{\mu}) \right). \tag{2.4}
\end{aligned}$$

The set of coefficients $\{c_i\}_{i=5,\dots,9}$ is determined by the Symanzik improvement program at one loop [19,22] (there is a discrepancy between the numerical values of c_6 reported in [22] and [19], we agree with the latter, see [23]). We shall often refer to these actions as standard action, 0-loop action and 1-loop action respectively.

The 1-loop action can be written in a more convenient form (mainly for the Monte Carlo simulation) after introducing the operators of Eq.(2.4) into Eq.(2.3),

$$\begin{aligned}
S^{1-\text{Symanzik}} &= - \sum_x \left\{ \frac{1}{g} \sum_{\mu} \left(\frac{4}{3} \vec{\phi}(x) \cdot \vec{\phi}(x + \hat{\mu}) - \frac{1}{12} \vec{\phi}(x) \cdot \vec{\phi}(x + 2\hat{\mu}) \right) + \right. \\
& c_5 \left[2 \sum_{\mu} \vec{\phi}(x) \cdot \vec{\phi}(x + 2\hat{\mu}) - 16 \sum_{\mu} \vec{\phi}(x) \cdot \vec{\phi}(x + \hat{\mu}) + \right. \\
& \quad \left. 2 \sum_{\mu \neq \nu} \vec{\phi}(x + \hat{\mu}) \cdot \vec{\phi}(x + \hat{\nu}) + 2 \sum_{\mu \neq \nu} \vec{\phi}(x) \cdot \vec{\phi}(x + \hat{\mu} + \hat{\nu}) \right] + \\
& c_6 \left[2 \sum_{\mu} \vec{\phi}(x) \cdot \vec{\phi}(x + 2\hat{\mu}) - 8 \sum_{\mu} \vec{\phi}(x) \cdot \vec{\phi}(x + \hat{\mu}) \right] + \\
& c_7 \left[-16 \sum_{\mu} \vec{\phi}(x) \cdot \vec{\phi}(x + \hat{\mu}) + 2 \sum_{\mu} \left(\vec{\phi}(x) \cdot \vec{\phi}(x + \hat{\mu}) \right)^2 + \right. \\
& \quad 2 \sum_{\mu} \left(\vec{\phi}(x) \cdot \vec{\phi}(x + \hat{\mu}) \right) \left(\vec{\phi}(x + \hat{\mu}) \cdot \vec{\phi}(x + 2\hat{\mu}) \right) + \\
& \quad \sum_{\mu \neq \nu} \left(\left(\vec{\phi}(x) \cdot \vec{\phi}(x + \hat{\mu}) \right) \left(\vec{\phi}(x) \cdot \vec{\phi}(x + \hat{\nu}) \right) + \left(\vec{\phi}(x) \cdot \vec{\phi}(x + \hat{\mu}) \right) \left(\vec{\phi}(x) \cdot \vec{\phi}(x - \hat{\nu}) \right) + \right. \\
& \quad \left. \left(\vec{\phi}(x) \cdot \vec{\phi}(x - \hat{\mu}) \right) \left(\vec{\phi}(x) \cdot \vec{\phi}(x + \hat{\nu}) \right) + \left(\vec{\phi}(x) \cdot \vec{\phi}(x - \hat{\mu}) \right) \left(\vec{\phi}(x) \cdot \vec{\phi}(x - \hat{\nu}) \right) \right) \Big] + \\
& c_8 \left[2 \sum_{\mu} \left(\vec{\phi}(x) \cdot \vec{\phi}(x + \hat{\mu}) \right)^2 - 8 \sum_{\mu} \vec{\phi}(x) \cdot \vec{\phi}(x + \hat{\mu}) + 2 \sum_{\mu} \left(\vec{\phi}(x) \cdot \vec{\phi}(x + \hat{\mu}) \right) \left(\vec{\phi}(x) \cdot \vec{\phi}(x - \hat{\mu}) \right) \right] + \\
& \frac{c_9}{16} \left[4 \sum_{\mu} \left(\vec{\phi}(x + \hat{\mu}) \cdot \vec{\phi}(x - \hat{\mu}) \right)^2 - 8 \sum_{\mu} \vec{\phi}(x) \cdot \vec{\phi}(x + 2\hat{\mu}) + \right. \\
& \quad \sum_{\mu \neq \nu} \left(\left(\vec{\phi}(x + \hat{\mu}) \cdot \vec{\phi}(x + \hat{\nu}) \right)^2 + \left(\vec{\phi}(x + \hat{\mu}) \cdot \vec{\phi}(x - \hat{\nu}) \right)^2 + \right. \\
& \quad \left. \left(\vec{\phi}(x - \hat{\mu}) \cdot \vec{\phi}(x + \hat{\nu}) \right)^2 + \left(\vec{\phi}(x - \hat{\mu}) \cdot \vec{\phi}(x - \hat{\nu}) \right)^2 \right) \Big] +
\end{aligned}$$

$$\begin{aligned}
& 2 \sum_{\mu \neq \nu} \left(\left(\vec{\phi}(x + \hat{\mu}) \cdot \vec{\phi}(x + \hat{\nu}) \right) \left(\vec{\phi}(x - \hat{\mu}) \cdot \vec{\phi}(x - \hat{\nu}) \right) + \right. \\
& \quad \left(\vec{\phi}(x + \hat{\mu}) \cdot \vec{\phi}(x - \hat{\nu}) \right) \left(\vec{\phi}(x - \hat{\mu}) \cdot \vec{\phi}(x + \hat{\nu}) \right) - \\
& \quad \left(\vec{\phi}(x + \hat{\mu}) \cdot \vec{\phi}(x + \hat{\nu}) \right) \left(\vec{\phi}(x + \hat{\mu}) \cdot \vec{\phi}(x - \hat{\nu}) \right) - \\
& \quad \left(\vec{\phi}(x + \hat{\mu}) \cdot \vec{\phi}(x + \hat{\nu}) \right) \left(\vec{\phi}(x - \hat{\mu}) \cdot \vec{\phi}(x + \hat{\nu}) \right) - \\
& \quad \left(\vec{\phi}(x - \hat{\mu}) \cdot \vec{\phi}(x - \hat{\nu}) \right) \left(\vec{\phi}(x + \hat{\mu}) \cdot \vec{\phi}(x - \hat{\nu}) \right) - \\
& \quad \left. \left(\vec{\phi}(x - \hat{\mu}) \cdot \vec{\phi}(x - \hat{\nu}) \right) \left(\vec{\phi}(x - \hat{\mu}) \cdot \vec{\phi}(x + \hat{\nu}) \right) \right) \Bigg\} . \tag{2.5}
\end{aligned}$$

We notice that this expression differs from the one shown in Table 2 of Ref. [24]. We can only say that we have carefully checked our Eq.(2.5) and that it coincides with the action given for example in Ref. [25].

The terms proportional to the coefficients c_i are of order $O(g)$. Therefore the 1-loop action is equal to the 0-loop action plus a sum of terms of higher order in g . This fact allows us to compute any perturbative quantity at k loops for the 1-loop action as the sum of the analogous quantity for the 0-loop action plus a set of diagrams with at most $(k - 1)$ loops.

To the actions showed in Eqs.(2.3) and (2.5) we must still add another term. The constraint shown in Eq.(2.2) is introduced under the path integral representation of the theory as a Dirac delta. Hence the partition function Z is written as

$$Z \equiv \int \mathcal{D}\vec{\phi}(x) \prod_x \delta \left(\left(\vec{\phi}(x) \right)^2 - 1 \right) e^{-S} , \tag{2.6}$$

where S indicates any of the lattice actions that we consider in the present paper. This constraint can be solved by rewriting the n -component field $\vec{\phi}$ in terms of a new $(n - 1)$ -component field $\vec{\pi}$ in the following way

$$\begin{aligned}
\vec{\phi}(x) &= \left(\phi_1(x), \dots, \phi_n(x) \right) \longrightarrow \left(\pi_1(x), \dots, \pi_{n-1}(x), \sqrt{1 - \pi_1(x)^2 - \dots - \pi_{n-1}(x)^2} \right) \\
&\equiv \left(\vec{\pi}(x), \sqrt{1 - \vec{\pi}(x)^2} \right) . \tag{2.7}
\end{aligned}$$

This transformation introduces a change of variables in the functional integration whose jacobian becomes a new term to be added to the original action as a measure action [26],

$$S \longrightarrow S + \sum_x \log \sqrt{1 - \vec{\pi}(x)^2} . \tag{2.8}$$

Perturbation theory is developed for the $(n - 1)$ -component field $\vec{\pi}(x)$ around the trivial configuration $\vec{\pi}(x) = 0$ for all x . The contribution to perturbative expansions from the measure term vanishes in the continuum when the divergences are dimensionally regularized [27].

In this paper we will compute the average density of action (or internal energy) E , up to fourth order in perturbation theory for all three actions on the lattice. We will begin by computing the free energy F to four loops, from which the internal energy reads

$$E = w_0 + \frac{1}{2} \frac{\partial}{\partial (1/g)} F , \quad F \equiv \lim_{L \rightarrow \infty} \frac{1}{L^2} \log Z , \tag{2.9}$$

where L^2 is the volume of a two-dimensional square lattice of side length L and Z is the partition function. The constant w_0 is determined in such a way that the operators representing E are (there is no summation in the index μ)

$$\begin{aligned}
E^{\text{standard}} &= \langle \vec{\phi}(0) \cdot \vec{\phi}(0 + \hat{\mu}) \rangle , \\
E^{0\text{-Symanzik}} &= \langle \frac{4}{3} \vec{\phi}(0) \cdot \vec{\phi}(0 + \hat{\mu}) - \frac{1}{12} \vec{\phi}(0) \cdot \vec{\phi}(0 + 2\hat{\mu}) \rangle , \\
E^{1\text{-Symanzik}} &= \langle \frac{4}{3} \vec{\phi}(0) \cdot \vec{\phi}(0 + \hat{\mu}) - \frac{1}{12} \vec{\phi}(0) \cdot \vec{\phi}(0 + 2\hat{\mu}) \rangle . \tag{2.10}
\end{aligned}$$

Then $w_0^{\text{standard}} = 1$ for the standard action and $w_0^{0\text{-Symanzik}} = w_0^{1\text{-Symanzik}} = 15/12$ for the two Symanzik actions.

In perturbation theory F is computed through the sum of the connected Feynman diagrams shown in Figs. 1–6. The analytic expressions of these diagrams are plagued with infrared (IR)–divergent integrals. An usual method to deal with these IR divergences is the introduction of an external source h coupled to the scalar field $\vec{\phi}$ [26,28]. We have chosen a different IR regulator: we have put the model into a square box of size L with periodic conditions on the boundaries. Then we have applied finite–size perturbation theory following Ref. [29]. This procedure has two consequences: firstly all zero modes can be excluded from the sums over momenta and secondly the action contains a new term which comes from a Faddeev–Popov determinant,

$$S_{FP} = -(n-1) \log \left(\sum_x \sqrt{1 - \vec{\pi}(x)^2} \right). \quad (2.11)$$

We have explicitly checked that the diagrams containing vertices from S_{FP} give rise to contributions which vanish in the thermodynamic limit $L \rightarrow \infty$ order by order up to four loops (individual diagrams give rise to non–vanishing terms but they cancel when adding up all diagrams at each order). We have not included this list of diagrams in Figs. 1–6 because in this paper we give the results for infinite–size lattices (in any case their contribution is rather small: for $L = 100$ it amounts to $O(10^{-5})$).

The diagrams in Figs. 1–6 lead to finite sums over internal momenta. Some of the sums become IR–divergent integrals in the limit $L \rightarrow \infty$. We have worked out these sums in order to separate their finite contribution from the IR–divergent one. In order to isolate these divergences we have performed only algebraic manipulations. We have avoided other kinds of manipulations, like applying derivatives to the integrands. Then the divergent pieces exactly cancel among themselves order by order. At the end, only the convergent sums remain (sums which in the limit $L \rightarrow \infty$ turn into IR–finite integrals) and we can safely remove the IR regulator by sending L to infinity. These finite integrals are then calculated as explained in Appendix C.

The calculation up to four loops for the standard action is known in the literature [30,4,31,5]. We have checked it obtaining analytical results in agreement. The calculation for the 0–loop Symanzik action is known only up to third order [32,5] and we have also successfully checked it. Therefore the new results of the present paper are the fourth order for the 0–loop Symanzik action and the full calculation to all orders, up to four loops, for the 1–loop improved Symanzik action. For completeness, in section 4 we will give the final results for all actions.

III. NOTATION AND IDENTITIES

In this section we will introduce some standard notation for the lattice perturbative calculations. Unless otherwise stated, we set the lattice spacing $a = 1$. The sine of half of the μ component of a momentum p is denoted by

$$\hat{p}_\mu \equiv 2 \sin \frac{p_\mu}{2}. \quad (3.1)$$

The inverse propagator in the standard action is $\hat{p}^2 \equiv \sum_\mu \hat{p}_\mu^2$. For the Symanzik action we introduce the notation

$$\square_p \equiv \sum_\mu \hat{p}_\mu^4, \quad (3.2)$$

and then the inverse Symanzik propagator, denoted by Π_p , reads

$$\begin{aligned} \Pi_p &\equiv \hat{p}^2 + \frac{1}{12} \square_p \equiv \sum_\mu \Pi_p^\mu, \\ \Pi_p^\mu &\equiv \hat{p}_\mu^2 + \frac{1}{12} \hat{p}_\mu^4. \end{aligned} \quad (3.3)$$

This propagator is the same for the two Symanzik actions.

Several identities among momenta are helpful to separate the finite and divergent contributions to E . One of these identities involve standard propagators [4],

$$\widehat{(p+q)}^2 + \widehat{(p+k)}^2 + \widehat{(p+r)}^2 = \hat{p}^2 + \hat{q}^2 + \hat{k}^2 + \hat{r}^2 - \Sigma_{pqkr}, \quad \Sigma_{pqkr} \equiv \sum_\mu \hat{p}_\mu \hat{q}_\mu \hat{k}_\mu \hat{r}_\mu, \quad (3.4)$$

and it is valid if $p + q + k + r = 0$. For the calculation using the Symanzik actions other relationships are needed, for instance [5],

$$\begin{aligned}\Pi_{p+q} + \Pi_{p+k} + \Pi_{p+r} &= \Pi_p + \Pi_q + \Pi_k + \Pi_r - \Sigma_{pqkr}^S, \\ \Sigma_{pqkr}^S &\equiv \frac{4}{3} \left(\sum_{\mu} \widehat{p}_{\mu} \widehat{q}_{\mu} \widehat{k}_{\mu} \widehat{r}_{\mu} - \sum_{\mu} \sin p_{\mu} \sin q_{\mu} \sin k_{\mu} \sin r_{\mu} \right),\end{aligned}\quad (3.5)$$

which again requires that $p + q + k + r = 0$. Moreover, in the calculation of tadpole diagrams we have used

$$\begin{aligned}\widehat{p+q}^2 &= \widehat{p}^2 + \widehat{q}^2 - \frac{1}{4} \widehat{p}^2 \widehat{q}^2 + \text{odd terms}, \\ (\widehat{p+q}^2)^2 &= (\widehat{p}^2)^2 + (\widehat{q}^2)^2 + \frac{1}{8} \left[(\widehat{p}^2)^2 (\widehat{q}^2)^2 - ((\widehat{p}^2)^2 - \square_p) \square_q - ((\widehat{q}^2)^2 - \square_q) \square_p \right] \\ &\quad + 2 \left(\widehat{p}^2 - \frac{1}{4} \square_p \right) \left(\widehat{q}^2 - \frac{1}{4} \square_q \right) + 2 \widehat{p}^2 \widehat{q}^2 - \frac{1}{2} (\widehat{p}^2)^2 \widehat{q}^2 - \frac{1}{2} (\widehat{q}^2)^2 \widehat{p}^2 + \text{odd terms}, \\ \Pi_{p+q} &= \Pi_p + \Pi_q - \frac{1}{12} \Pi_p \square_q - \frac{1}{12} \Pi_q \square_p + \frac{5}{144} \square_p \square_q + \text{odd terms}, \\ \square_{p+q} &= \square_p + \square_q - \frac{5}{4} \Pi_p \square_q - \frac{5}{4} \Pi_q \square_p + 3 \Pi_p \Pi_q + \frac{7}{16} \square_p \square_q + \text{odd terms}.\end{aligned}\quad (3.6)$$

The Feynman diagrams necessary for the calculation are shown in Figures 1–6.

Let us show two partial calculations as examples of the procedure we have followed. On finite lattices of side length L any component of a momentum p can take L discrete values, for instance the first component $p_1 = 2\pi\ell_1/L$ ($\ell_1 = 0, 1, \dots, L-1$). Therefore the sums over momenta are

$$\frac{1}{L} \sum_{\ell_1=0}^{L-1} \frac{1}{L} \sum_{\ell_2=0}^{L-1}^*, \quad (3.7)$$

and become integrals in the limit $L \rightarrow \infty$,

$$\int_{-\pi}^{+\pi} \frac{d^2 p}{(2\pi)^2}. \quad (3.8)$$

In Eq.(3.7) the zero mode $\ell_1 = \ell_2 = 0$ must be excluded as prescribed in Ref. [29]. This is the meaning of the stars in Eq.(3.7). In the following we use the shorthand

$$\sum_p, \quad (3.9)$$

to denote the sum in Eq.(3.7) when it sums over the momentum p . Besides, although at finite L we are dealing with discrete sums, we will often call them “integrals” and the expression summed will often be named “integrand”.

Momentum conservation is expressed through a Kronecker delta, $L^2 \delta^2(p + q + k + \dots)$. p, q, k, \dots are momenta which satisfy $p_1 = 2\pi\ell_1/L$, etc. The argument of the delta function is assumed to be periodic modulus 2π . The sum in Eq.(3.9), or in Eq.(3.7), acting on a delta function leads to

$$\sum_p L^2 \delta^2(p + q + k + \dots) = 1 - \frac{1}{L^2} L^2 \delta^2(q + k + \dots). \quad (3.10)$$

The additional $O(1/L^2)$ contribution in the r.h.s. of this equation can produce terms like

$$\frac{1}{L^2} \sum_p \frac{1}{(\Pi_p)^2}, \quad \frac{1}{L^2} \sum_p \sum_q \sum_k \frac{1}{\Pi_p \Pi_q \Pi_k} L^2 \delta^2(p + q + k), \quad (3.11)$$

which are finite after the removal of the IR-regulator. These terms are exclusive of the finite size L regularization: notice for example that they cannot be expressed as usual integrals, neither after the $L \rightarrow \infty$ limit. We have checked that such terms cancel out when we sum up the contributions from all diagrams at each order.

Terms like those in Eq.(3.11), and similar ones coming from the Faddeev–Popov action (2.11), yield finite contributions to the final result in the 1D $O(n)$ model [29]. In the 2D model such terms cancel out at least up to fourth order.

To the diagram 6 of Fig 6 it contributes the coefficient c_9 of the 1-loop improved action through the vertex

$$\begin{aligned} \frac{1}{2} c_9 \sum_p \sum_q \sum_k \sum_r \sum_s \sum_t L^2 \delta^2(p+q+k+r+s+t) \\ \times \sum_{\mu\nu} \sum_{abc} \pi^a(p) \pi^a(q) \pi^b(k) \pi^b(r) \pi^c(s) \pi^c(t) \sin p_\mu \sin q_\nu \sin(k+r)_\mu \sin(s+t)_\nu, \end{aligned} \quad (3.12)$$

where p, q, k, r, s and t are the momenta which satisfy $p_1 = 2\pi\ell_1/L$, etc. and a, b and c the $O(n)$ indices (running from 1 to $n-1$). To simplify the notation we use the same symbol to indicate the field in coordinate space $\vec{\pi}(x)$ and in momentum space $\vec{\pi}(p)$. The contraction of the six legs to produce three tadpoles can be done in three topologically non equivalent ways and leads to

$$\begin{aligned} c_9 g^3 \sum_{\mu\nu} \left\{ (n-1)^2 \sum_p \sum_q \sum_k \frac{\sin p_\mu \sin p_\nu \sin(q+k)_\mu \sin(q+k)_\nu}{\Pi_p \Pi_q \Pi_k} \right. \\ + 2(n-1) \sum_p \sum_q \sum_k \frac{\sin p_\mu \sin q_\nu \sin(q+k)_\mu \sin(p+k)_\nu}{\Pi_p \Pi_q \Pi_k} \\ \left. + 2(n-1) \sum_p \sum_q \sum_k \frac{\sin p_\mu \sin q_\nu \sin(p+k)_\mu \sin(q+k)_\nu}{\Pi_p \Pi_q \Pi_k} \right\}. \end{aligned} \quad (3.13)$$

The first integral is non zero only if $\mu = \nu$ and, by using $\sin^2 p_\mu = \hat{p}_\mu^2 - 1/4 \hat{p}_\mu^4$ and the first and last identities in Eq.(3.6), gives

$$c_9 g^3 \frac{1}{2} (n-1)^2 \left(1 - \frac{1}{3} Y_1\right)^2 \left(2Z_1 - 1 + \frac{1}{3} Y_1\right) + O\left(\frac{\log L}{L^2}\right), \quad (3.14)$$

where Y_i in the limit $L \rightarrow \infty$ are finite integrals defined and calculated in Appendix A and Z_i are the above-mentioned sums which in the thermodynamic limit diverge and that at the end of the calculation must disappear,

$$Z_i \equiv \sum_p \left(\frac{1}{\Pi_p}\right)^i. \quad (3.15)$$

Notice that this expression is well defined as long as L is finite because the zero mode $\Pi_p = 0$ is missing. It becomes an ill-defined integral only in the thermodynamic limit. For large L we have that $Z_1 \sim \log L$, $Z_2 \sim L^2$, $Z_3 \sim L^4$, etc.

The second integration in Eq.(3.13) yields, after some algebra and taking into account again that it vanishes when $\mu \neq \nu$,

$$\begin{aligned} 2 c_9 g^3 (n-1) \sum_p \sum_q \sum_k \frac{\sum_\mu \sin^2 p_\mu \sin^2 q_\mu \sin^2 k_\mu}{\Pi_p \Pi_q \Pi_k} \\ = c_9 g^3 \frac{1}{2} (n-1) \left(1 - \frac{1}{3} Y_1\right)^2 \left(2Z_1 - 1 + \frac{1}{3} Y_1\right) + O\left(\frac{\log L}{L^2}\right). \end{aligned} \quad (3.16)$$

Finally the third integral in Eq.(3.13) needs some algebra to eliminate several odd terms like $\sum_p \sin p_\mu / \Pi_p$. After this work, it can be rewritten as

$$\begin{aligned} 2 c_9 g^3 (n-1) \sum_p \sum_q \sum_k \frac{\sum_{\mu\nu} \sin^2 p_\mu \sin^2 q_\nu \cos k_\mu \cos k_\nu}{\Pi_p \Pi_q \Pi_k} \\ = c_9 g^3 \frac{1}{2} (n-1) \left(1 - \frac{1}{3} Y_1\right)^2 \left(4Z_1 - \frac{5}{4} + \frac{1}{6} Y_1 + \frac{1}{576} Y_{2,1}\right) + O\left(\frac{\log L}{L^2}\right), \end{aligned} \quad (3.17)$$

where the property $\cos k_\mu = 1 - 1/2 \hat{k}_\mu^2$ was used.

The second example of calculation is taken from the coefficient of c_9 in the diagram 5 of Fig. 6. The vertices with four legs are

$$\begin{aligned} \frac{1}{16} c_9 \sum_p \sum_q \sum_k \sum_r L^2 \delta^2(p+q+k+r) \\ \times \sum_{ab} \pi^a(p) \pi^a(q) \pi^b(k) \pi^b(r) \left[\left(\widehat{p+k}\right)^2 + \left(\widehat{p-k}\right)^2 \left(\widehat{q-r}\right)^2 - 2 \left(\widehat{p+k}\right)^2 \left(\widehat{q-r}\right)^2 \right], \end{aligned} \quad (3.18)$$

from the black spot and

$$-\frac{1}{8g} \sum_p \sum_q \sum_k \sum_r L^2 \delta^2(p+q+k+r) \times \sum_{ab} \pi^a(p) \pi^a(q) \pi^b(k) \pi^b(r) \Pi_{p+q} , \quad (3.19)$$

which is the vertex coming from the 0-loop Symanzik part of the action. The contraction gives

$$-g^3 \frac{1}{16} c_9 \left[(n-1)^2 I_1 + (n-1) (I_1 + I_2) \right] , \quad (3.20)$$

where

$$\begin{aligned} I_1 &\equiv \sum_p \sum_q \sum_k \sum_r L^2 \delta^2(p+q+k+r) \\ &\quad \times \frac{\Pi_{p+q} \left[\left(\widehat{p+k}^2 \right)^2 + \left(\widehat{p-k}^2 \right) \left(\widehat{q-r}^2 \right) - 2 \left(\widehat{p+k}^2 \right) \left(\widehat{q-r}^2 \right) \right]}{\Pi_p \Pi_q \Pi_k \Pi_r} \\ I_2 &\equiv \sum_p \sum_q \sum_k \sum_r L^2 \delta^2(p+q+k+r) \\ &\quad \times \frac{\Pi_{p+q} \left[\left(\widehat{p+q}^2 \right)^2 + \left(\widehat{p-q}^2 \right) \left(\widehat{k-r}^2 \right) - 2 \left(\widehat{p+q}^2 \right) \left(\widehat{k-r}^2 \right) \right]}{\Pi_p \Pi_q \Pi_k \Pi_r} . \end{aligned} \quad (3.21)$$

The numerators can be easily worked out. For instance, the square brackets in the numerator of I_1 can be rewritten by using the Kronecker delta in Eq.(3.21) ($\Delta_{p,q}$ is introduced in Appendix A),

$$\begin{aligned} &\left(\widehat{p+k}^2 \right)^2 + \left(\widehat{p-k}^2 \right) \left(\widehat{q-r}^2 \right) - 2 \left(\widehat{p+k}^2 \right) \left(\widehat{q-r}^2 \right) \\ &= \left(\Delta_{p,k} + \widehat{p}^2 + \widehat{k}^2 \right) \left(\Delta_{q,r} + \widehat{q}^2 + \widehat{r}^2 \right) \\ &\quad + \left(\Delta_{p,-k} + \widehat{p}^2 + \widehat{k}^2 \right) \left(\Delta_{q,-r} + \widehat{q}^2 + \widehat{r}^2 \right) \\ &\quad - 2 \left(\Delta_{p,k} + \widehat{p}^2 + \widehat{k}^2 \right) \left(\Delta_{q,-r} + \widehat{q}^2 + \widehat{r}^2 \right) \\ &= \Delta_{p,k} \Delta_{q,r} + \Delta_{p,-k} \Delta_{q,-r} - 2 \Delta_{p,k} \Delta_{q,-r} , \end{aligned} \quad (3.22)$$

which is true under the integration. After taking the limit $L \rightarrow \infty$, this expression leads immediately to the final result for $I_1 = S_{11} + S_{14} - 2 S_{15}$. Analogously $I_2 = S_{10} + S_{16} - 2 S_{17}$. The integrals S_i are defined and evaluated in Appendix A.

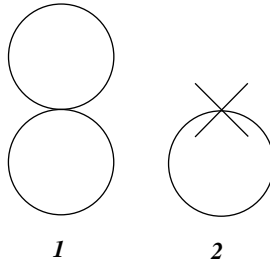


FIG. 1. Feynman diagrams contributing to the free energy of the standard and 0-loop Symanzik actions at two loops. The lines represent the propagation of the scalar field $\vec{\pi}$, the cross stands for a vertex from the measure action.

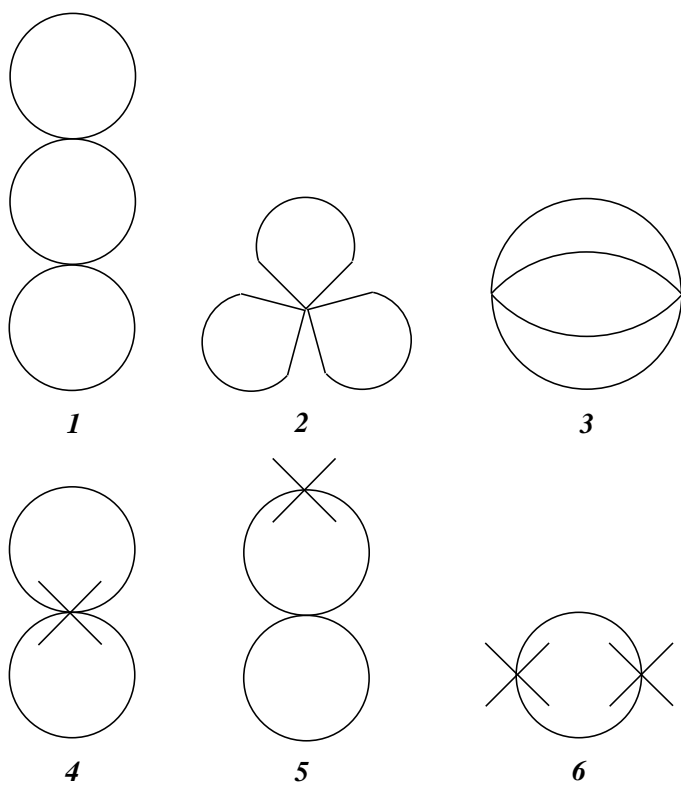


FIG. 2. The same as Fig. 1 at three loops.

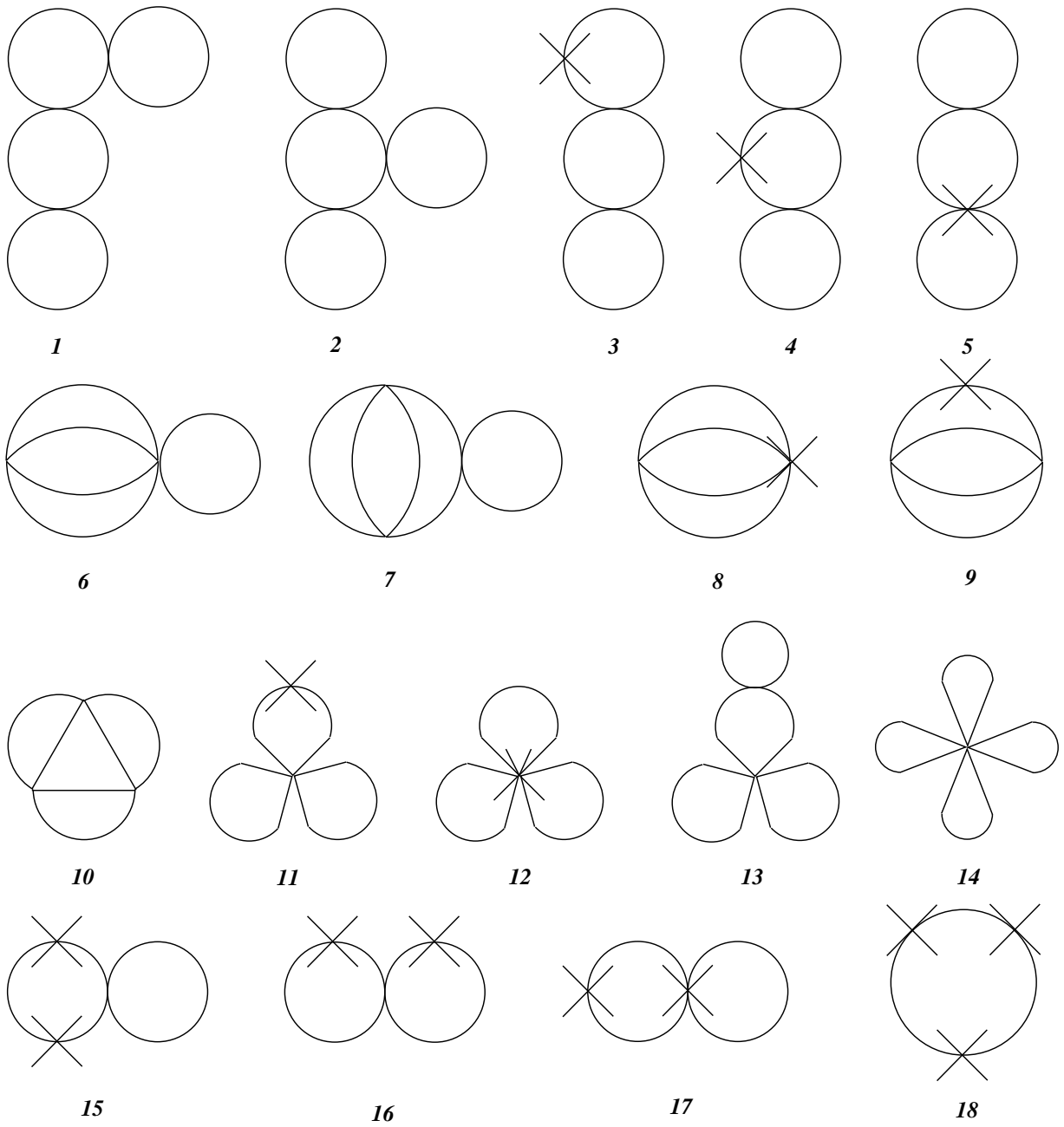


FIG. 3. The same as Fig. 1 at four loops.

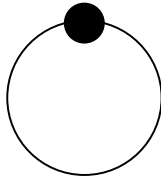


FIG. 4. The Feynman diagrams necessary for the calculation of the free energy at two loops for the 1-loop Symanzik action are those shown in Fig. 1 plus the diagram displayed in this figure. The black spot indicates a vertex proportional to some coefficient c_i .

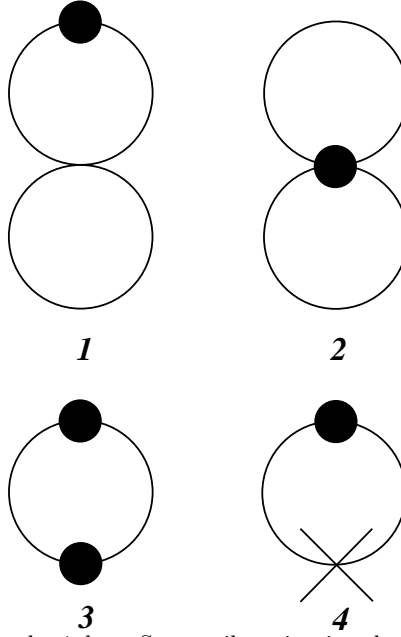


FIG. 5. The free energy at three loops for the 1-loop Symanzik action is calculated by adding up the diagrams of Fig. 2 to those displayed in this figure. Same notation as in Fig. 4.

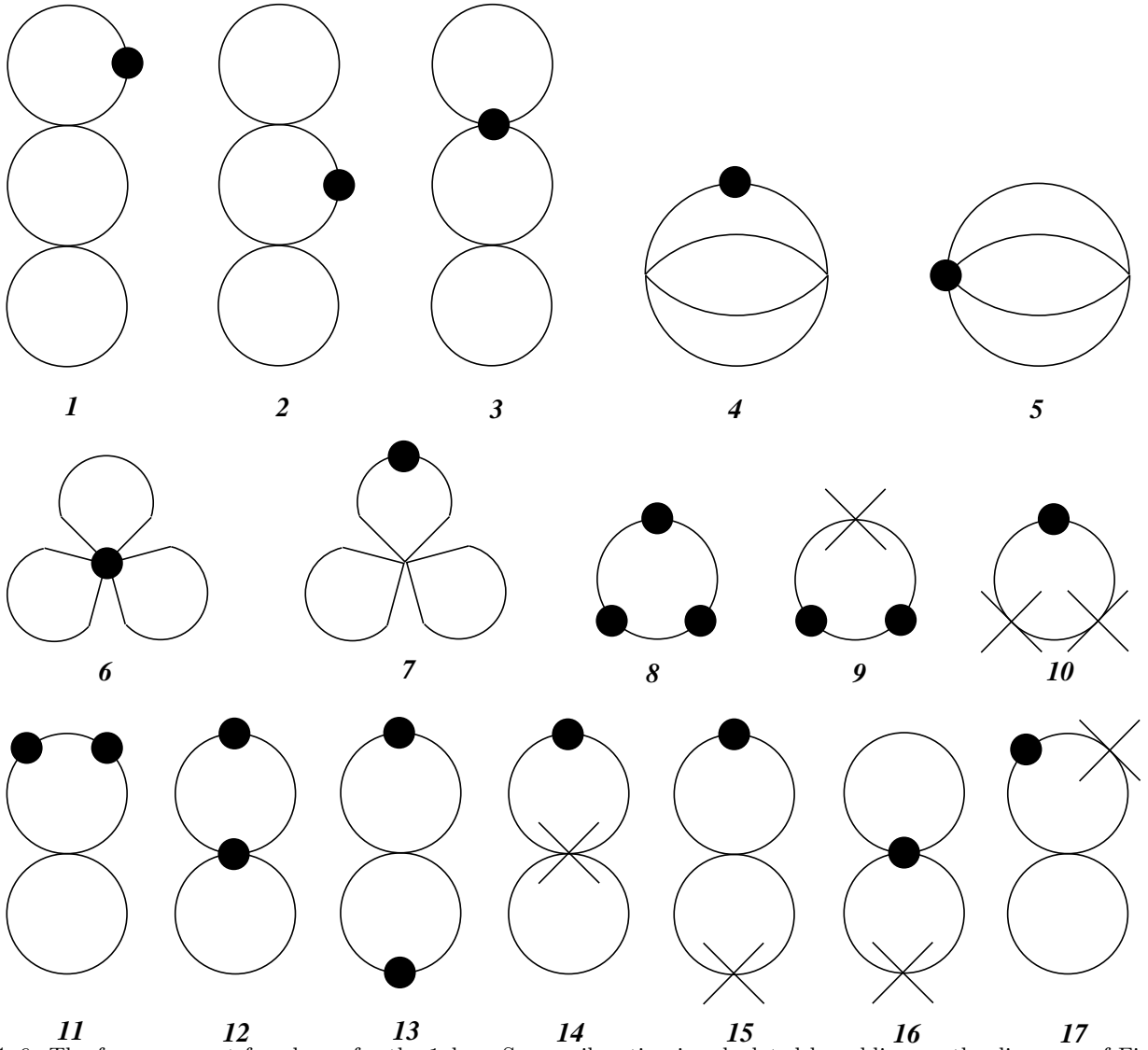


FIG. 6. The free energy at four loops for the 1-loop Symanzik action is calculated by adding up the diagrams of Fig. 3 to those displayed in this figure. Same notation as in Fig. 4.

IV. RESULTS FOR E AND EFFECTIVE SCHEME

In this section we will give the results for the internal energy E and will explain how to obtain the coefficients for β^L in the corresponding effective scheme. We will show the results for all three actions. Those for the standard action are not new [30,4,31,5] but we have checked all of them. We write the perturbative expansion of E as

$$E = w_0 - w_1 g - w_2 g^2 - w_3 g^3 - w_4 g^4 - \dots . \quad (4.1)$$

Then,

$$\begin{aligned} w_0^{\text{standard}} &= 1 , \\ w_1^{\text{standard}} &= \frac{(n-1)}{4} , \\ w_2^{\text{standard}} &= \frac{(n-1)}{32} , \\ w_3^{\text{standard}} &= \frac{(n-1)^2}{16} K + \frac{(n-1)}{16} \left(\frac{1}{6} - K + \frac{1}{3} J \right) , \end{aligned}$$

$$\begin{aligned}
w_4^{\text{standard}} = & \frac{3}{8} (n-1) \left(\frac{1}{128} - \frac{1}{2} H_1 - \frac{1}{4} H_2 - \frac{1}{3} H_3 + \frac{1}{24} J - \frac{1}{8} K - \frac{1}{4} H_5 \right) \\
& + \frac{3}{8} (n-1)^2 \left(\frac{1}{256} + \frac{1}{2} H_1 + \frac{1}{4} H_2 + \frac{1}{3} H_3 + \frac{1}{12} H_4 + \frac{1}{8} K + \frac{1}{3} H_5 \right) \\
& - \frac{(n-1)^3}{32} H_5 .
\end{aligned} \tag{4.2}$$

The integrals K , J and H_i for $i = 1, \dots, 5$ are defined and calculated in [4,5]. They are shown in our Appendix A.

The results for the 0-loop Symanzik action are known up to three loops [32,5] and here we have added the fourth order. The first terms $w_0^{0\text{-Symanzik}}$, ..., $w_3^{0\text{-Symanzik}}$ have been checked obtaining full agreement. The complete set of coefficients is

$$\begin{aligned}
w_0^{0\text{-Symanzik}} &= \frac{15}{12} , \\
w_1^{0\text{-Symanzik}} &= \frac{(n-1)}{4} , \\
w_2^{0\text{-Symanzik}} &= \frac{(n-1)}{48} Y_1 \left(1 - \frac{5}{24} Y_1 \right) , \\
w_3^{0\text{-Symanzik}} &= \frac{(n-1)^2}{16} K^S + \frac{(n-1)}{16} \left(\frac{1}{3} - K^S + \frac{1}{3} J^S + \frac{1}{36} Y_2 \right. \\
&\quad \left. - Y_1 \left(\frac{5}{12} + \frac{5}{216} Y_2 \right) + Y_1^2 \left(\frac{11}{48} + \frac{25}{5184} Y_2 \right) - \frac{205}{5184} Y_1^3 \right) , \\
w_4^{0\text{-Symanzik}} &= (n-1) \left(-\frac{5}{256} - \frac{1}{96} S_9 - \frac{3}{16} H_1^S - \frac{3}{32} H_2^S - \frac{1}{8} H_3^S + \frac{1}{16} J^S - \frac{1}{8} \overline{J^S} + \frac{1}{16} \widetilde{J^S} \right. \\
&\quad - \frac{3}{16} K^S + \frac{3}{8} \overline{K^S} - \frac{3}{16} \widetilde{K^S} - \frac{3}{32} H_5^S + \frac{35}{768} Y_1 + \frac{5}{1152} S_9 Y_1 - \frac{1}{48} J^S Y_1 \\
&\quad + \frac{5}{96} \overline{J^S} Y_1 - \frac{5}{192} \widetilde{J^S} Y_1 + \frac{1}{16} K^S Y_1 - \frac{5}{32} \overline{K^S} Y_1 + \frac{5}{64} \widetilde{K^S} Y_1 - \frac{75}{2048} Y_1^2 \\
&\quad + \frac{53}{4096} Y_1^3 - \frac{9005}{5308416} Y_1^4 - \frac{5}{1536} Y_2 + \frac{91}{18432} Y_1 Y_2 - \frac{535}{221184} Y_1^2 Y_2 \\
&\quad + \frac{1025}{2654208} Y_1^3 Y_2 - \frac{5}{55296} Y_2^2 + \frac{25}{331776} Y_1 Y_2^2 - \frac{125}{7962624} Y_1^2 Y_2^2 \\
&\quad \left. + \frac{1}{6912} Y_3 - \frac{5}{27648} Y_1 Y_3 + \frac{25}{331776} Y_1^2 Y_3 - \frac{125}{11943936} Y_1^3 Y_3 \right) \\
&\quad + (n-1)^2 \left(\frac{5}{512} + \frac{1}{96} S_9 + \frac{3}{16} H_1^S + \frac{3}{32} H_2^S + \frac{1}{8} H_3^S + \frac{3}{16} K^S - \frac{3}{8} \overline{K^S} + \frac{3}{16} \widetilde{K^S} \right. \\
&\quad + \frac{1}{32} H_4^S + \frac{1}{8} H_5^S - \frac{25}{1536} Y_1 - \frac{5}{1152} S_9 Y_1 - \frac{1}{16} K^S Y_1 + \frac{5}{32} \overline{K^S} Y_1 \\
&\quad \left. - \frac{5}{64} \widetilde{K^S} Y_1 + \frac{125}{12288} Y_1^2 - \frac{593}{221184} Y_1^3 + \frac{2725}{10616832} Y_1^4 \right) \\
&\quad - \frac{(n-1)^3}{32} H_5^S .
\end{aligned} \tag{4.3}$$

The integrals K^S , J^S , $\overline{K^S}$, $\overline{J^S}$, $\widetilde{K^S}$, $\widetilde{J^S}$, Y_1 , Y_2 and Y_3 were introduced in [5]. They are shown for completeness in Appendix A. The new integrals are S_9 , H_1^S , H_2^S , H_3^S , H_4^S and H_5^S . All of them are listed and evaluated in the Appendix.

The expression of the internal energy for the 1-loop Symanzik improved action is a new result of the present paper for all loops and is written in terms of the following coefficients

$$w_0^{1\text{-Symanzik}} = w_0^{0\text{-Symanzik}} ,$$

$$\begin{aligned}
w_1^{1-\text{Symanzik}} &= w_1^{0-\text{Symanzik}} , \\
w_2^{1-\text{Symanzik}} &= w_2^{0-\text{Symanzik}} + \frac{(n-1)}{2} \left(c_5 \left(3 + \frac{1}{144} Y_{2,1} \right) + c_6 Y_1 \right) , \\
w_3^{1-\text{Symanzik}} &= w_3^{0-\text{Symanzik}} + (n-1)^2 \left(\left(c_7 + \frac{1}{2} c_8 \right) \left(1 - \frac{1}{6} Y_1 + \frac{1}{144} Y_1^2 \right) + c_9 \left(\frac{1}{2} - \frac{1}{3} Y_1 + \frac{1}{18} Y_1^2 \right) \right) \\
&\quad + (n-1) \left(c_9 \left(\frac{3}{2} - Y_1 + \frac{1}{6} Y_1^2 \right) + c_8 \left(1 - \frac{2}{3} Y_1 + \frac{25}{144} Y_1^2 \right) \right. \\
&\quad \left. + c_7 \left(\frac{25}{16} - \frac{25}{24} Y_1 + \frac{17}{72} Y_1^2 + \frac{1}{384} Y_{2,1} - \frac{1}{1152} Y_1 Y_{2,1} + \frac{1}{331776} (Y_{2,1})^2 \right) \right. \\
&\quad \left. + c_6^2 Y_2 + c_6 \left(\frac{3}{2} - \frac{5}{4} Y_1 + \frac{29}{96} Y_1^2 + \frac{1}{12} Y_2 - \frac{5}{144} Y_1 Y_2 \right) \right. \\
&\quad \left. + c_5 \left(\frac{33}{16} - \frac{5}{4} Y_1 + \frac{35}{144} Y_1^2 - \frac{17}{1152} Y_{2,1} + \frac{5}{864} Y_1 Y_{2,1} + \frac{1}{331776} (Y_{2,1})^2 \right) \right. \\
&\quad \left. + \frac{1}{1728} Y_{3,2} - \frac{5}{20736} Y_1 Y_{3,2} \right) + c_5 c_6 \left(24 - \frac{1}{3} Y_{2,1} + \frac{1}{72} Y_{3,2} \right) \\
&\quad \left. + c_5^2 \left(\frac{55}{4} - \frac{1}{432} Y_{3,1} + \frac{1}{20736} Y_{4,2} \right) \right) . \tag{4.4}
\end{aligned}$$

The analytical form of $w_4^{1-\text{Symanzik}}$ is very lengthy and its explicit expression is reported in Appendix D.

Numerically these expansions read (we only report five significant digits, which are usually enough for simulation purposes, although the results shown in Appendix A for the finite integrals can be readily calculated with greater precision)

$$\begin{aligned}
E^{\text{standard}} &= 1 - \frac{n-1}{4} g - \frac{n-1}{32} g^2 - \left[0.0072699 (n-1) + 0.0059930 (n-1)^2 \right] g^3 \\
&\quad - \left[0.0028167 (n-1) + 0.0034299 (n-1)^2 + 0.0015673 (n-1)^3 \right] g^4 , \\
E^{0-\text{Symanzik}} &= \frac{15}{12} - \frac{n-1}{4} g - 0.024449 (n-1) g^2 - \left[0.0044905 (n-1) + 0.0042082 (n-1)^2 \right] g^3 \\
&\quad - \left[0.0014508 (n-1) + 0.0017541 (n-1)^2 + 0.0010241 (n-1)^3 \right] g^4 , \\
E^{1-\text{Symanzik}} &= \frac{15}{12} - \frac{n-1}{4} g - \left[(0.024449 + 1.60443 c_5 + 1.02179 c_6) (n-1) \right] g^2 \\
&\quad - \left[(0.0044905 + 12.5286 c_5^2 + 0.26629 c_6 + 4.7831 c_6^2 + 0.44431 c_5 + 15.0482 c_5 c_6 \right. \\
&\quad \left. + 0.44752 c_7 + 0.36265 c_8 + 0.15246 c_9) (n-1) \right. \\
&\quad \left. + (0.0042082 + 0.68841 c_7 + 0.34420 c_8 + 0.050819 c_9) (n-1)^2 \right] g^3 \\
&\quad - \left[(0.0014508 + 0.11759 c_5 + 4.7668 c_5^2 + 108.382 c_5^3 + 0.068884 c_6 + 5.8854 c_5 c_6 \right. \\
&\quad \left. + 186.142 c_5^2 c_6 + 1.7585 c_6^2 + 111.566 c_5 c_6^2 + 23.6332 c_6^3 + 0.11683 c_7 \right. \\
&\quad \left. + 9.2935 c_5 c_7 + 5.7226 c_6 c_7 + 0.078923 c_8 + 7.1066 c_5 c_8 + 4.5248 c_6 c_8 \right. \\
&\quad \left. - 0.031694 c_9 + 2.0109 c_5 c_9 + 1.2889 c_6 c_9) (n-1) \right. \\
&\quad \left. + (0.0017541 + 0.068786 c_5 + 0.047639 c_6 + 0.15515 c_7 + 12.853 c_5 c_7 \right. \\
&\quad \left. + 8.18909 c_6 c_7 + 0.069749 c_8 + 6.4265 c_5 c_8 + 4.0946 c_6 c_8 \right. \\
&\quad \left. - 0.011464 c_9 + 0.67029 c_5 c_9 + 0.42964 c_6 c_9) (n-1)^2 \right. \\
&\quad \left. + 0.0010241 (n-1)^3 \right] g^4 . \tag{4.5}
\end{aligned}$$

Notice that at order $O(g)$ the result is the same in all three cases. This is a consequence of the equipartition of the energy. The numerical value of the quartic coefficient in E^{standard} differs roughly by 3% from the result in Ref. [5].

This is due to the more accurate determination of the integral H_3 obtained in the present paper (see Appendix A). This difference is too small to change at all any conclusion of Ref. [5].

Now the energy-based effective coupling constant g_E is defined in a non-perturbative way,

$$g_E \equiv \frac{w_0 - E^{\text{MC}}}{w_1} , \quad (4.6)$$

where E^{MC} is the Monte Carlo measured value of the internal energy at some value of the bare coupling g . The perturbative expansion of g_E in terms of this coupling g is obtained from the previously calculated expansion for E

$$g_E = g + \frac{w_2}{w_1} g^2 + \frac{w_3}{w_1} g^3 + \frac{w_4}{w_1} g^4 + \dots . \quad (4.7)$$

The lattice beta function β^L in terms of the bare coupling g is [20]

$$\beta^L(g) \equiv -a \frac{dg}{da} = -\beta_0 g^2 - \beta_1 g^3 - \beta_2 g^4 - \beta_3 g^5 - \dots , \quad (4.8)$$

and in terms of g_E it becomes

$$\begin{aligned} \beta^L(g_E) &= \beta^L(g(g_E)) \frac{dg_E}{dg}(g_E) \\ &= -\beta_0 g_E^2 - \beta_1 g_E^3 - \frac{\beta_2 w_1^2 - \beta_1 w_1 w_2 - \beta_0 (w_2^2 - w_1 w_3)}{w_1^2} g_E^4 \\ &\quad - \frac{\beta_3 w_1^3 - 2\beta_2 w_1^2 w_2 + \beta_1 w_1 w_2^2 + 2\beta_0 (2w_2^3 - 3w_1 w_2 w_3 + w_1^2 w_4)}{w_1^3} g_E^5 - \dots , \end{aligned} \quad (4.9)$$

where the function $g(g_E)$ is obtained by inverting Eq.(4.7).

The integration of the beta function yields the dependence of the lattice spacing a on the coupling constant

$$a\Lambda = (\beta_0 g)^{-\beta_1/\beta_0^2} \exp\left(-\frac{1}{\beta_0 g}\right) (1 + O(g)) , \quad (4.10)$$

where Λ is the integration constant, the lattice Lambda parameter. An analogous equation can be derived in the effective scheme with an integration constant Λ_E . From Eq.(4.7) and Eq.(4.10) the ratio of these two constants can be exactly determined,

$$\Lambda_E = \Lambda \exp\left\{\frac{w_2}{w_1 \beta_0}\right\} . \quad (4.11)$$

V. CONCLUSIONS

We have calculated the perturbative expansion of the internal energy for the tree-level and 1-loop improved Symanzik actions on the lattice for the 2D nonlinear σ -model with symmetry $O(n)$ up to fourth order in the coupling constant. These results are shown in Eq.(4.5). The definitions that we have adopted for the internal energy E are shown in Eq.(2.10). These expansions allow the definition of an effective coupling g_E . We expect that the series which determine the lattice spacing in terms of the coupling constant are better behaved if expressed in powers of g_E . This hope will be checked in a future publication where we plan to calculate the mass gap of the model through a Monte Carlo simulation. To this end, we have also given the analytic expression of the lattice beta function in terms of g_E .

The calculation is rather involved and it has been done separately by the two authors. Only the final results were compared. Also the numerical value of the integrals in Appendix A has been obtained independently and checked afterwards. Further checks were done: for example some subsets of diagrams must altogether yield an IR-finite result. These tests come out when we consider other definitions for the energy operator different from the one shown in Eq.(2.10). For instance if we use the whole expression of the 1-loop Symanzik action as an energy operator, Eq.(2.5), then diagrams 9, 11, 12 and 13 of Figure 6 are multiplied by a different factor from the rest of diagrams. Then this subset, taken separately, must produce an IR-finite result.

Acknowledgements

It is a pleasure to thank Paolo Butera, Marco Comi and Giuseppe Marchesini for useful discussions, Andrea Pelissetto for clarifying comments on the IR regularization and Paolo Butera for a critical reading of the manuscript.

VI. APPENDIX A

List of finite integrals

We will give the list of finite integrals that we have used to express the results for E as a manifestly finite quantity. The basic notation is introduced in section 3. Besides, the following definitions will be needed:

$$\begin{aligned}
\Delta_{p,q} &\equiv (\widehat{p+q})^2 - \widehat{p}^2 - \widehat{q}^2, \\
\Delta_{p,-q} &\equiv (\widehat{p-q})^2 - \widehat{p}^2 - \widehat{q}^2, \\
\Delta_{p,q}^S &\equiv \Pi_{p+q} - \Pi_p - \Pi_q, \\
\Delta_{p,-q}^S &\equiv \Pi_{p-q} - \Pi_p - \Pi_q, \\
\Delta_{p,q}^\square &\equiv \square_{p+q} - \square_p - \square_q, \\
\Delta_{p,q}^\mu &\equiv (\widehat{p+q})_\mu^2 - \widehat{p}_\mu^2 - \widehat{q}_\mu^2.
\end{aligned} \tag{6.1}$$

In the three-loop integrals we use the notation

$$\int D_3 \equiv \int_{-\pi}^{+\pi} \frac{d^2 p}{(2\pi)^2} \int_{-\pi}^{+\pi} \frac{d^2 q}{(2\pi)^2} \int_{-\pi}^{+\pi} \frac{d^2 k}{(2\pi)^2} \int_{-\pi}^{+\pi} \frac{d^2 r}{(2\pi)^2} (2\pi)^2 \delta^2(p+q+k+r), \tag{6.2}$$

and the definitions and numerical values of the integrals are

$$K \equiv \int D_3 \frac{\Delta_{p,q} \Delta_{k,r}}{\widehat{p}^2 \widehat{q}^2 \widehat{k}^2 \widehat{r}^2} = 0.0958876, \tag{6.3}$$

$$J \equiv \int D_3 \frac{(\Sigma_{pqkr})^2}{\widehat{p}^2 \widehat{q}^2 \widehat{k}^2 \widehat{r}^2} = 0.136620, \tag{6.4}$$

$$K^S \equiv \int D_3 \frac{\Delta_{p,q}^S \Delta_{k,r}^S}{\Pi_p \Pi_q \Pi_k \Pi_r} = 0.0673313, \tag{6.5}$$

$$J^S \equiv \int D_3 \frac{(\Sigma_{pqkr}^S)^2}{\Pi_p \Pi_q \Pi_k \Pi_r} = 0.104551, \tag{6.6}$$

$$\overline{K^S} \equiv \int D_3 \frac{\Delta_{p,q}^S \Delta_{k,r}^S \widehat{p}^2}{(\Pi_p)^2 \Pi_q \Pi_k \Pi_r} = 0.0572726, \tag{6.7}$$

$$\overline{J^S} \equiv \int D_3 \frac{(\Sigma_{pqkr}^S)^2 \widehat{p}^2}{(\Pi_p)^2 \Pi_q \Pi_k \Pi_r} = 0.0867807, \tag{6.8}$$

$$\widetilde{K^S} \equiv \int D_3 \frac{\Delta_{p,q}^S \Delta_{k,r}}{\Pi_p \Pi_q \Pi_k \Pi_r} = 0.0578002, \tag{6.9}$$

$$\widetilde{J^S} \equiv \int D_3 \frac{\Sigma_{pqkr}^S \Sigma_{pqkr}}{\Pi_p \Pi_q \Pi_k \Pi_r} = 0.0809553, \quad (6.10)$$

$$S_1 \equiv \int D_3 \frac{\Delta_{p,q}^S (\Delta_{k,r})^2}{\Pi_p \Pi_q \Pi_k \Pi_r} = -0.283407, \quad (6.11)$$

$$S_2 \equiv \int D_3 \frac{\Delta_{p,q}^S \Delta_{k,r} \widehat{k}^2}{\Pi_p \Pi_q \Pi_k \Pi_r} = 0.184636, \quad (6.12)$$

$$S_3 \equiv \int D_3 \frac{\Delta_{p,q}^S \widehat{k}^2 \widehat{r}^2}{\Pi_p \Pi_q \Pi_k \Pi_r} = -0.134904, \quad (6.13)$$

$$S_4 \equiv \int D_3 \frac{\Sigma_{pqkr}^S (\widehat{p}^2)^2}{\Pi_p \Pi_q \Pi_k \Pi_r} = 0.148440, \quad (6.14)$$

$$S_5 \equiv \int D_3 \frac{\Sigma_{pqkr}^S \Delta_{p,q} \Delta_{k,r}}{\Pi_p \Pi_q \Pi_k \Pi_r} = 0.261935, \quad (6.15)$$

$$S_6 \equiv \int D_3 \frac{\Sigma_{pqkr}^S \Delta_{p,q} \widehat{k}^2}{\Pi_p \Pi_q \Pi_k \Pi_r} = -0.181963, \quad (6.16)$$

$$S_7 \equiv \int D_3 \frac{\Sigma_{pqkr}^S \widehat{k}^2 \widehat{r}^2}{\Pi_p \Pi_q \Pi_k \Pi_r} = 0.124096, \quad (6.17)$$

$$S_8 \equiv \int D_3 \frac{\Delta_{p,q}^S \Delta_{k,r}^\square}{\Pi_p \Pi_q \Pi_k \Pi_r} = 0.114374, \quad (6.18)$$

$$S_9 \equiv \int D_3 \frac{\Sigma_{pqkr}^S \square_p}{\Pi_p \Pi_q \Pi_k \Pi_r} = 0.0891356, \quad (6.19)$$

$$S_{10} \equiv \int D_3 \frac{\Pi_{p+q} \Delta_{p,q} \Delta_{k,r}}{\Pi_p \Pi_q \Pi_k \Pi_r} = 0.0881094, \quad (6.20)$$

$$S_{11} \equiv \int D_3 \frac{\Pi_{p+q} \Delta_{p,k} \Delta_{q,r}}{\Pi_p \Pi_q \Pi_k \Pi_r} = 0.215207, \quad (6.21)$$

$$S_{12} \equiv \int D_3 \frac{\Pi_{p+q} \sum_\mu \Delta_{p,q}^\mu \Delta_{k,r}^\mu}{\Pi_p \Pi_q \Pi_k \Pi_r} = 0.0649188, \quad (6.22)$$

$$S_{13} \equiv \int D_3 \frac{\Pi_{p+q} \sum_\mu \Delta_{p,k}^\mu \Delta_{q,r}^\mu}{\Pi_p \Pi_q \Pi_k \Pi_r} = 0.113020, \quad (6.23)$$

$$S_{14} \equiv \int D_3 \frac{\Pi_{p+q} \Delta_{p,-k} \Delta_{q,-r}}{\Pi_p \Pi_q \Pi_k \Pi_r} = 0.170225, \quad (6.24)$$

$$S_{15} \equiv \int D_3 \frac{\Pi_{p+q} \Delta_{p,k} \Delta_{q,-r}}{\Pi_p \Pi_q \Pi_k \Pi_r} = 0.108536 , \quad (6.25)$$

$$S_{16} \equiv \int D_3 \frac{\Pi_{p+q} \Delta_{p,-q} \Delta_{k,-r}}{\Pi_p \Pi_q \Pi_k \Pi_r} = 0.233959 , \quad (6.26)$$

$$S_{17} \equiv \int D_3 \frac{\Pi_{p+q} \Delta_{p,q} \Delta_{k,-r}}{\Pi_p \Pi_q \Pi_k \Pi_r} = 0.0963352 , \quad (6.27)$$

$$S_{18} \equiv \int D_3 \frac{\Delta_{p,q}^S \Delta_{k,r}^S (\hat{p}^2)^2}{(\Pi_p)^2 \Pi_q \Pi_k \Pi_r} = 0.184379 , \quad (6.28)$$

$$S_{19} \equiv \int D_3 \frac{\Delta_{p,q}^S \Delta_{k,r}^S \square_p}{(\Pi_p)^2 \Pi_q \Pi_k \Pi_r} = 0.120705 , \quad (6.29)$$

$$S_{20} \equiv \int D_3 \frac{(\Sigma_{pqkr}^S)^2 (\hat{p}^2)^2}{(\Pi_p)^2 \Pi_q \Pi_k \Pi_r} = 0.335775 , \quad (6.30)$$

$$S_{21} \equiv \int D_3 \frac{(\Sigma_{pqkr}^S)^2 \square_p}{(\Pi_p)^2 \Pi_q \Pi_k \Pi_r} = 0.213243 . \quad (6.31)$$

The above listed integrals are not all independent. In Appendix B we give a few identities that these integrals satisfy.

The measure for the four-loop integrals is

$$\begin{aligned} \int D_4 \equiv & \int_{-\pi}^{+\pi} \frac{d^2 p}{(2\pi)^2} \int_{-\pi}^{+\pi} \frac{d^2 q}{(2\pi)^2} \int_{-\pi}^{+\pi} \frac{d^2 k}{(2\pi)^2} \int_{-\pi}^{+\pi} \frac{d^2 r}{(2\pi)^2} \int_{-\pi}^{+\pi} \frac{d^2 s}{(2\pi)^2} \int_{-\pi}^{+\pi} \frac{d^2 t}{(2\pi)^2} \\ & \times (2\pi)^2 \delta^2(p+q+k+r) (2\pi)^2 \delta^2(k+r+s+t) , \end{aligned} \quad (6.32)$$

and the numerical values of the four-loop integrals are (the result of H_3 differs in the third significant digit from the less accurate result given in [5])

$$H_1 \equiv \int D_4 \frac{\Delta_{p,q} \Delta_{k,r} \Sigma_{pqst}}{\hat{p}^2 \hat{q}^2 \hat{k}^2 \hat{r}^2 \hat{s}^2 \hat{t}^2} = 0.0378134 , \quad (6.33)$$

$$H_2 \equiv \int D_4 \frac{\Delta_{k,r} \Sigma_{pqkr} \Sigma_{pqst}}{\hat{p}^2 \hat{q}^2 \hat{k}^2 \hat{r}^2 \hat{s}^2 \hat{t}^2} = -0.0322778 , \quad (6.34)$$

$$H_3 \equiv \int D_4 \frac{\Delta_{p,k} \Delta_{r,s} \Delta_{q,-t}}{\hat{p}^2 \hat{q}^2 \hat{k}^2 \hat{r}^2 \hat{s}^2 \hat{t}^2} = -0.0128736 , \quad (6.35)$$

$$H_4 \equiv \int D_4 \frac{\Sigma_{pqkr} \Sigma_{krst} \Sigma_{pqst}}{\hat{p}^2 \hat{q}^2 \hat{k}^2 \hat{r}^2 \hat{s}^2 \hat{t}^2} = 0.0411085 , \quad (6.36)$$

$$H_5 \equiv \int D_4 \frac{\Delta_{p,q} \Delta_{k,r} \Delta_{s,t}}{\hat{p}^2 \hat{q}^2 \hat{k}^2 \hat{r}^2 \hat{s}^2 \hat{t}^2} = -0.0501531 , \quad (6.37)$$

$$H_1^S \equiv \int D_4 \frac{\Delta_{p,q}^S \Delta_{k,r}^S \Sigma_{pqst}^S}{\Pi_p \Pi_q \Pi_k \Pi_r \Pi_s \Pi_t} = 0.0218345 , \quad (6.38)$$

$$H_2^S \equiv \int D_4 \frac{\Delta_{k,r}^S \Sigma_{pqkr}^S \Sigma_{pqst}^S}{\Pi_p \Pi_q \Pi_k \Pi_r \Pi_s \Pi_t} = -0.0181139 , \quad (6.39)$$

$$H_3^S \equiv \int D_4 \frac{\Delta_{p,k}^S \Delta_{r,s}^S \Delta_{q,-t}^S}{\Pi_p \Pi_q \Pi_k \Pi_r \Pi_s \Pi_t} = -0.0042338 , \quad (6.40)$$

$$H_4^S \equiv \int D_4 \frac{\Sigma_{pqkr}^S \Sigma_{krst}^S \Sigma_{pqst}^S}{\Pi_p \Pi_q \Pi_k \Pi_r \Pi_s \Pi_t} = 0.0262036 , \quad (6.41)$$

$$H_5^S \equiv \int D_4 \frac{\Delta_{p,q}^S \Delta_{k,r}^S \Delta_{s,t}^S}{\Pi_p \Pi_q \Pi_k \Pi_r \Pi_s \Pi_t} = -0.0327709 . \quad (6.42)$$

On the other hand the one-loop integrals are defined as

$$Y_i \equiv \int_{-\pi}^{+\pi} \frac{d^2 q}{(2\pi)^2} \left(\frac{\square_q}{\Pi_q} \right)^i , \quad Y_{i,j} \equiv \int_{-\pi}^{+\pi} \frac{d^2 q}{(2\pi)^2} \frac{(\square_q)^i}{(\Pi_q)^j} , \quad (2i \geq j) , \quad (6.43)$$

and their results are listed in Table 1. Notice that $Y_{i,i} \equiv Y_i$.

Table 1: One-loop integrals.

Y_1	2.0435764382979844236
Y_2	4.7830710733439886212
Y_3	11.816615246907788250
$Y_{1,2}$	0.4729502261432961899
$Y_{2,1}$	30.077096804291341057
$Y_{3,1}$	558.65986413777280387
$Y_{3,2}$	77.324121011413132160
$Y_{4,1}$	11817.841483609309517
$Y_{4,2}$	1489.1480965521674895
$Y_{4,3}$	202.26364872706189510
$Y_{5,2}$	32200.496224041766111
$Y_{5,3}$	4006.2729031961906982
$Y_{6,3}$	88276.902118545681915

Some of the one-loop integrals were introduced in Ref. [5]. All $Y_{i,j}$ with $i \neq j$ show up only in the results for the 1-loop Symanzik action.

In the perturbative expansion of the 1-loop Symanzik action there appear some vertices with a very high mass dimension. Once these vertices are inserted in the corresponding Feynman diagrams, all propagators may cancel leading to non-fractional integrands. In this case the following expression can be useful

$$\int_{-\pi}^{+\pi} \frac{dq}{2\pi} (\hat{q}_\mu)^{2m} = 2 \binom{2m-1}{m} , \quad m \geq 1 . \quad (6.44)$$

For example,

$$\int_{-\pi}^{+\pi} \frac{d^2 q}{(2\pi)^2} \hat{q}^2 = 4 , \quad \int_{-\pi}^{+\pi} \frac{d^2 q}{(2\pi)^2} \square_q = 12 , \quad \int_{-\pi}^{+\pi} \frac{d^2 q}{(2\pi)^2} \Pi_q = 5 . \quad (6.45)$$

VII. APPENDIX B

Identities among the integrals

Some of the integrals S_i in the above Appendix are actually related among themselves. We have not taken advantage of these relationships in the final expressions of section 4 because we are not sure to have discovered all of them. In this Appendix we show the identities that we have found out,

$$\frac{1}{12}S_8 = K^S - \widetilde{K^S}, \quad \frac{1}{12}S_{19} = K^S - \overline{K^S}, \quad \frac{1}{12}S_{21} = J^S - \overline{J^S}, \quad (7.1)$$

$$2S_{11} + S_{10} + S_5 = \frac{29}{8} - \frac{35 Y_1^3}{864} + Y_1^2 \left(\frac{65}{96} + \frac{5 Y_{2,1}}{41472} \right) + \frac{Y_{2,1}}{576} + \frac{(Y_{2,1})^2}{165888} - Y_1 \left(\frac{83}{32} + \frac{11 Y_{2,1}}{6912} + \frac{(Y_{2,1})^2}{1990656} \right), \quad (7.2)$$

$$S_1 - S_{10} - 4S_3 - 2S_7 + \frac{2}{3}S_4 = -\frac{67}{32} + \frac{41 Y_1^3}{432} + Y_1^2 \left(-\frac{169}{192} + \frac{7 Y_{2,1}}{82944} \right) - \frac{7 Y_{2,1}}{4608} - \frac{(Y_{2,1})^2}{663552} + \frac{(Y_{2,1})^3}{286654464} + Y_1 \left(\frac{39}{16} + \frac{Y_{2,1}}{1728} - \frac{(Y_{2,1})^2}{995328} \right). \quad (7.3)$$

We give the proof of the relationship Eq.(7.2). Twice the numerator in S_{11} plus the numerator in S_{10} is equal to

$$\Delta_{p,q} \Delta_{k,r} (\Pi_{p+k} + \Pi_{p+q} + \Pi_{p+r}), \quad (7.4)$$

which, by using Eq.(3.5) under the integration, becomes

$$\Delta_{p,q} \Delta_{k,r} (4 \Pi_p - \Sigma_{pqkr}^S). \quad (7.5)$$

Therefore the following expression is true

$$2S_{11} + S_{10} = 4 \int D_3 \frac{\Delta_{p,q} \Delta_{k,r}}{\Pi_q \Pi_k \Pi_r} - S_5. \quad (7.6)$$

The numerator in the integrand can be rewritten after some straightforward algebra under the integration

$$\Delta_{p,q} \Delta_{k,r} = -\frac{1}{2} (\widehat{k^2})^2 \widehat{r^2} + \frac{1}{4} \widehat{q^2} (\widehat{k+r^2})^2 + \frac{1}{8} (\widehat{q^2})^2 \widehat{k^2} \widehat{r^2}, \quad (7.7)$$

and now a use of the second identity in Eq.(3.6) yields the final result shown in Eq.(7.2).

VIII. APPENDIX C

Methods for the numerical calculation of the integrals

We have used three methods to calculate the various finite integrals listed in Appendix A. They are: *i*) an extrapolation to infinite size of the results obtained at small sizes, *ii*) the Gauss integration and *iii*) an extension of the coordinate space method [33] with the Symanzik propagators. Several integrals were evaluated by using more than one method for checking purposes. In this Appendix we will briefly describe methods *i*) and *iii*).

In the first method we calculated the integral I at small lattices L obtaining $I(L)$ for several L . Then we extrapolated this set of results to infinite size. The extrapolating formula was

$$I(L) = I(L = \infty) + \frac{b_1}{L^m} + \frac{b_2 \log L}{L^m}, \quad (8.1)$$

where the correct result for the integral is $I(L = \infty)$. The exponent in the denominators is $m = 1$ for the one-loop integrals and $m = 2$ for all other integrals. The errors were determined by looking at the stability of the figures

against the increase of L . In fact such a stability indicates that further terms in the expansion Eq. (8.1) are irrelevant for the last stable digit. For one-loop integrals we obtained actually many significant digits by working up to lattice sizes as large as $L = 10000$. For the S_i integrals the largest size was $L = 60$.

For finite L the integral $I(L)$ is actually a sum of terms. In this sum we have excluded the momenta which lead to vanishing propagators ($\hat{p}^2 = 0$ or $\Pi_p = 0$). In the limit $L \rightarrow \infty$ this procedure implies the exclusion of a region that has zero measure in the corresponding integration and therefore it has no consequences on the final extrapolated result $I(L = \infty)$. We have checked this statement by repeating the calculation on several one loop integrals firstly *i*) by excluding only the zero mode $\hat{p}^2 = 0$ and secondly *ii*) by excluding the lines $\hat{p}_1^2 = 0$ or $\hat{p}_2^2 = 0$. Both methods yielded exactly the same final extrapolated number.

The integrals H_i and H_i^S emerge in the calculation of the diagram 10 of Fig. 3. They are four-loop integrals and as a consequence a direct application of the above method is rather slow and a poor precision is obtained. In order to simplify the evaluation of these four-loop integrals, one can take advantage of the topology of the diagram and rewrite them as an effective two-loop integral

$$H_i = \int_{-\pi}^{+\pi} \frac{d^2 q}{(2\pi)^2} \sum_k \left(\prod_{j=1}^3 \int_{-\pi}^{+\pi} \frac{d^2 p}{(2\pi)^2} \frac{\mathcal{N}_i^{kj}(p, q-p)}{\widehat{p}^2 (q-p)^2} \right), \quad (8.2)$$

and analogously for H_i^S . The sum \sum_k over numerators $\mathcal{N}_i^{kj}(p, q-p)$ contains one term in H_5 and 64 terms in H_3^S . Nevertheless, in all cases the integration of Eq.(8.2) is much faster than a direct integration of Eqs.(6.33)–(6.42). The largest lattice size used for the evaluation of these integrals with the method of Eq.(8.2) was $L = 400$.

In some cases we have also applied the coordinate space method to check our results. We need to extend this method, introduced in [33,34], to include the case of improved propagators. Here we will show with some detail the calculation of J^S . This requires the previous evaluation of the improved free propagator $G(x)$

$$G(x) \equiv \int_{-\pi}^{+\pi} \frac{d^2 p}{(2\pi)^2} \frac{e^{i p x} - 1}{\Pi_p}. \quad (8.3)$$

In terms of $G(x)$ the integral J^S reads

$$J^S = \frac{1}{9} \sum_x \left(16 \sum_{\mu\nu} (\partial_\mu^+ \partial_\nu^+ G(x))^4 - 2 \sum_{\mu\nu} (\partial_\nu^+ (\partial_\mu^+ + \partial_\mu^-) G(x))^4 + \frac{1}{16} \sum_{\mu\nu} ((\partial_\nu^+ + \partial_\nu^-) (\partial_\mu^+ + \partial_\mu^-) G(x))^4 \right), \quad (8.4)$$

where ∂^\pm have been defined in Eq.(2.4). The procedure is analogous to what is done for the standard case, the only new features arise because of the presence of an improved Symanzik propagator. This propagator, Eq.(8.3), satisfies

$$\mathcal{L} G(x) = -\delta^2(x), \quad (8.5)$$

where $\delta^2(x)$ is the Kronecker function on a two-dimensional lattice and \mathcal{L} is the improved Laplacian

$$\mathcal{L} \equiv \frac{4}{3} \sum_\mu \partial_\mu^+ \partial_\mu^- - \frac{1}{12} \sum_\mu (\partial_\mu^+ + \partial_\mu^-)^2. \quad (8.6)$$

If $x \neq 0$ Eq.(8.5) provides

$$\sum_\mu G(x + 2\hat{\mu}) = - \sum_\mu G(x - 2\hat{\mu}) + 4G(x) + 16 \sum_\mu [G(x + \hat{\mu}) + G(x - \hat{\mu}) - 2G(x)]. \quad (8.7)$$

Now, by using

$$\frac{\partial}{\partial p_\mu} \log \Pi_p = \frac{1}{\Pi_p} \left(\frac{8}{3} \sin p_\mu - \frac{1}{3} \sin 2p_\mu \right), \quad (8.8)$$

we obtain

$$\frac{4}{3} (G(x + \hat{\mu}) - G(x - \hat{\mu})) - \frac{1}{6} (G(x + 2\hat{\mu}) - G(x - 2\hat{\mu})) = x_\mu X^S(x), \quad (8.9)$$

where

$$X^S(x) \equiv \int_{-\pi}^{+\pi} \frac{d^2 p}{(2\pi)^2} e^{i p x} \log \Pi_p . \quad (8.10)$$

Eq.(8.9) provides a recurrence relation for the propagator,

$$G(x + 2\hat{\mu}) = G(x - 2\hat{\mu}) + 8 [G(x + \hat{\mu}) - G(x - \hat{\mu})] - 6x_\mu X^S(x) \quad (8.11)$$

which must be complemented with the relationship

$$X^S(x) = \frac{1}{\sum_\mu x_\mu} \left(10 G(x) + \sum_\mu \left(\frac{1}{3} G(x - 2\hat{\mu}) - 4 G(x - \hat{\mu}) - \frac{4}{3} G(x + \hat{\mu}) \right) \right) , \quad (8.12)$$

obtained from Eq.(8.7) and (8.9) after summing over μ .

From the recurrence Eqs.(8.11)–(8.12) and the symmetry relations $G(x_1, x_2) = G(x_2, x_1)$, $G(x_1, x_2) = G(-x_1, x_2)$ (x_1 and x_2 are the two components of the coordinate site x) we can determine $G(x)$ for any x in terms of the values of $G(x)$ at a basic set of 10 sites. These sites are shown in Fig. 7. Below we give the values of $G(x)$ on these points in terms of $Y_{i,j}$ and Y_i

$$\begin{aligned} G(0, 0) &= 0, \\ G(1, 0) &= \frac{1}{48} Y_1 - \frac{1}{4}, \\ G(1, 1) &= \frac{1}{1152} Y_{2,1} - \frac{1}{12} Y_1 - \frac{1}{8}, \\ G(2, 0) &= \frac{1}{3} Y_1 - 1, \\ G(2, 1) &= \frac{1}{27648} Y_{3,1} - \frac{1}{576} Y_{2,1} - \frac{7}{48} Y_1 - \frac{5}{576}, \\ G(2, 2) &= \frac{1}{331776} Y_{4,1} - \frac{1}{1728} Y_{3,1} + \frac{5}{144} Y_{2,1} - \frac{4}{3} Y_1 + \frac{917}{576}, \\ G(3, 0) &= \frac{-1}{13824} Y_{3,1} + \frac{1}{32} Y_{2,1} + \frac{27}{16} Y_1 - \frac{1363}{288}, \\ G(3, 1) &= \frac{1}{1728} Y_{3,1} - \frac{7}{128} Y_{2,1} + \frac{7}{12} Y_1 - \frac{19}{72}, \\ G(3, 2) &= \frac{1}{7962624} Y_{5,1} - \frac{1}{55296} Y_{4,1} + \frac{7}{27648} Y_{3,1} + \frac{1}{64} Y_{2,1} - \frac{119}{48} Y_1 + \frac{233341}{55296}, \\ G(3, 3) &= \frac{1}{95551488} Y_{6,1} - \frac{1}{331776} Y_{5,1} + \frac{1}{3072} Y_{4,1} - \frac{17}{864} Y_{3,1} + \frac{89}{128} Y_{2,1} - \frac{27}{4} Y_1 + \frac{169681}{497664}. \end{aligned} \quad (8.13)$$

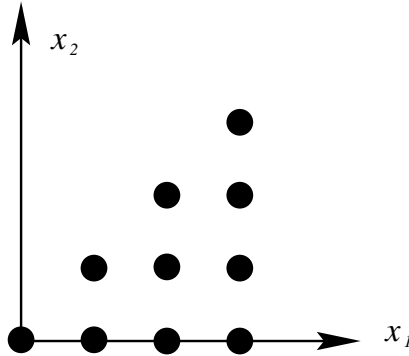


FIG. 7. The basic lattice sites for the evaluation of the propagator in the coordinate space method.

We must know the values of G on the above set of sites very accurately but the above results for $Y_{i,j}$ are not precise enough (see Table 1). This goal is achieved by using the recurrence Eqs.(8.11)–(8.12) backwards.

In some cases, we also need the calculation of the squared propagator $G_2(x)$ defined as

$$G_2(x) \equiv \int_{-\pi}^{+\pi} \frac{d^2 p}{(2\pi)^2} \frac{e^{i p x} - 1 + \frac{1}{2} \sum_{\mu} \Pi_p^{\mu} x_{\mu}^2}{(\Pi_p)^2}. \quad (8.14)$$

The quantity Π_p^{μ} has been defined in Eq.(3.3). The laplacian Eq.(8.6) applied on G_2 produces $\mathcal{L}G_2(x) = -G(x)$, or alternatively

$$\sum_{\mu} G_2(x + 2\hat{\mu}) = - \sum_{\mu} G_2(x - 2\hat{\mu}) + 4G_2(x) + 16 \sum_{\mu} [G_2(x + \hat{\mu}) + G_2(x - \hat{\mu}) - 2G_2(x)] + 12 G(x). \quad (8.15)$$

Now, by using that

$$\frac{\partial}{\partial p_{\mu}} \frac{1}{\Pi_p} = - \frac{1}{(\Pi_p)^2} \left(\frac{8}{3} \sin p_{\mu} - \frac{1}{3} \sin 2p_{\mu} \right), \quad (8.16)$$

we obtain the basic recurrence relation

$$G_2(x + 2\hat{\mu}) = G_2(x - 2\hat{\mu}) + 8 [G_2(x + \hat{\mu}) - G_2(x - \hat{\mu})] + 6x_{\mu} X_2^S(x), \quad (8.17)$$

where $X_2^S(x) = G(x) + 1/4\pi$. Summing this equation over μ and using Eq.(8.15), we get

$$X_2^S(x) = \frac{1}{\sum_{\mu} x_{\mu}} \left(2 G(x) - 10 G_2(x) - \frac{1}{3} \sum_{\mu} G_2(x - 2\hat{\mu}) + \frac{4}{3} \sum_{\mu} G_2(x + \hat{\mu}) + 4 \sum_{\mu} G_2(x - \hat{\mu}) \right), \quad (8.18)$$

which must be complemented with Eq.(8.17) to obtain $G_2(x)$ for any x from the same basic set of points shown in Fig. 7.

Further tricks necessary for the computation of the integrals are analogous to those already explained in Ref. [33].

IX. APPENDIX D

Form of $w_4^{1-\text{Symanzik}}$

The expression for $w_4^{1-\text{Symanzik}}$ can be written as the sum of $w_4^{0-\text{Symanzik}}$ plus a pure 1-loop Symanzik action contribution. This contribution contains terms proportional to $(n-1)$, $(n-1)^2$ and several powers of the coefficients c_i , $i = 5, \dots, 9$. Let us parametrize these contributions in the following way

$$\begin{aligned} w_4^{1-\text{Symanzik}} = w_4^{0-\text{Symanzik}} &+ \frac{3}{2}(n-1)^2 \left(c_5 q_5 + c_6 q_6 + c_7 q_7 + c_8 q_8 + c_9 q_9 + c_5 c_7 q_{57} + c_5 c_8 q_{58} \right. \\ &\quad \left. + c_5 c_9 q_{59} + c_6 c_7 q_{67} + c_6 c_8 q_{68} + c_6 c_9 q_{69} \right) \\ &+ \frac{3}{2}(n-1) \left(c_5 p_5 + c_5^2 p_{55} + c_5^3 p_{555} + c_6 p_6 + c_6^2 p_{66} + c_6^3 p_{666} \right. \\ &\quad \left. + c_5 c_6 p_{56} + c_5^2 c_6 p_{556} + c_5 c_6^2 p_{566} + c_7 p_7 + c_8 p_8 + c_9 p_9 \right. \\ &\quad \left. + c_5 c_7 p_{57} + c_5 c_8 p_{58} + c_5 c_9 p_{59} + c_6 c_7 p_{67} + c_6 c_8 p_{68} + c_6 c_9 p_{69} \right). \quad (9.1) \end{aligned}$$

Now the several coefficients q_i are

$$\begin{aligned} q_5 &= \frac{1}{2} S_{18} - \frac{1}{4} S_1 - S_2 - \frac{1}{2} S_3, \\ q_6 &= \frac{1}{2} S_{19} - \frac{1}{4} S_8, \\ q_7 &= -\frac{1}{4} - \frac{1}{4} S_{10} + \frac{19}{48} Y_1 - \frac{67}{576} Y_1^2 + \frac{49}{6912} Y_1^3 - \frac{1}{72} Y_2 + \frac{1}{144} Y_1 Y_2 - \frac{5}{10368} Y_1^2 Y_2, \\ q_8 &= -\frac{1}{8} - \frac{1}{4} S_{12} + \frac{19}{96} Y_1 - \frac{67}{1152} Y_1^2 + \frac{49}{13824} Y_1^3 - \frac{1}{144} Y_2 + \frac{1}{288} Y_1 Y_2 - \frac{5}{20736} Y_1^2 Y_2, \\ q_9 &= -\frac{1}{2} - \frac{1}{16} (S_{11} + S_{14} - 2S_{15}) + \frac{2}{3} Y_1 - \frac{41}{144} Y_1^2 + \frac{17}{432} Y_1^3 - \frac{1}{36} Y_2 + \frac{1}{48} Y_1 Y_2 - \frac{5}{1296} Y_1^2 Y_2, \end{aligned}$$

$$\begin{aligned}
q_{57} &= 8 - \frac{2}{3}Y_1 + \frac{1}{12}Y_{2,1} - \frac{1}{144}Y_1 Y_{2,1} - \frac{1}{432}Y_{3,2} + \frac{1}{5184}Y_1 Y_{3,2}, \\
q_{58} &= 4 - \frac{1}{3}Y_1 + \frac{1}{24}Y_{2,1} - \frac{1}{288}Y_1 Y_{2,1} - \frac{1}{864}Y_{3,2} + \frac{1}{10368}Y_1 Y_{3,2}, \\
q_{59} &= -2 + \frac{2}{3}Y_1 + \frac{1}{8}Y_{2,1} - \frac{1}{24}Y_1 Y_{2,1} - \frac{1}{216}Y_{3,2} + \frac{1}{648}Y_1 Y_{3,2}, \\
q_{67} &= 4Y_1 - \frac{1}{3}Y_1^2 - \frac{1}{3}Y_2 + \frac{1}{36}Y_1 Y_2, \\
q_{68} &= 2Y_1 - \frac{1}{6}Y_1^2 - \frac{1}{6}Y_2 + \frac{1}{72}Y_1 Y_2, \\
q_{69} &= 2Y_1 - \frac{2}{3}Y_1^2 - \frac{2}{3}Y_2 + \frac{2}{9}Y_1 Y_2,
\end{aligned} \tag{9.2}$$

and those p_i are

$$\begin{aligned}
p_5 &= -\frac{123}{128} + \frac{1}{6}S_{20} - \frac{1}{2}S_{18} + \frac{1}{4}S_1 + S_2 + \frac{1}{2}S_3 - \frac{1}{3}S_4 + \frac{1}{4}S_5 + S_6 + S_7 + \frac{93}{64}Y_1 - \frac{185}{288}Y_1^2 + \frac{215}{2304}Y_1^3 \\
&\quad + \frac{353}{18432}Y_{2,1} - \frac{121}{6912}Y_1 Y_{2,1} + \frac{655}{165888}Y_1^2 Y_{2,1} - \frac{11}{884736}(Y_{2,1})^2 + \frac{7}{1327104}Y_1 (Y_{2,1})^2 + \frac{1}{1146617856}(Y_{2,1})^3 \\
&\quad - \frac{5}{48}Y_2 + \frac{145}{1728}Y_1 Y_2 - \frac{175}{10368}Y_1^2 Y_2 + \frac{5}{10368}Y_2 Y_{2,1} - \frac{25}{124416}Y_1 Y_2 Y_{2,1} - \frac{11}{6912}Y_{3,2} \\
&\quad + \frac{77}{55296}Y_1 Y_{3,2} - \frac{605}{1990656}Y_1^2 Y_{3,2} + \frac{1}{1990656}Y_{2,1} Y_{3,2} - \frac{5}{23887872}Y_1 Y_{2,1} Y_{3,2} \\
&\quad - \frac{5}{248832}Y_2 Y_{3,2} + \frac{25}{2985984}Y_1 Y_2 Y_{3,2} + \frac{1}{20736}Y_{4,3} - \frac{5}{124416}Y_1 Y_{4,3} + \frac{25}{2985984}Y_1^2 Y_{4,3}, \\
p_{55} &= -\frac{557}{48} + \frac{425}{72}Y_1 + \frac{349}{768}Y_{2,1} - \frac{25}{144}Y_1 Y_{2,1} - \frac{7}{6912}(Y_{2,1})^2 + \frac{13}{1728}Y_{3,1} - \frac{5}{1728}Y_1 Y_{3,1} \\
&\quad - \frac{1}{248832}Y_{2,1} Y_{3,1} - \frac{5}{288}Y_{3,2} + \frac{35}{5184}Y_1 Y_{3,2} + \frac{5}{62208}Y_{2,1} Y_{3,2} - \frac{5}{2985984}(Y_{3,2})^2 - \frac{11}{27648}Y_{4,2} \\
&\quad + \frac{5}{31104}Y_1 Y_{4,2} + \frac{1}{11943936}Y_{2,1} Y_{4,2} + \frac{1}{124416}Y_{5,3} - \frac{5}{1492992}Y_1 Y_{5,3}, \\
p_{555} &= \frac{1670}{27} + \frac{5}{5184}Y_{4,1} - \frac{1}{31104}Y_{5,2} + \frac{1}{2239488}Y_{6,3}, \\
p_6 &= -\frac{5}{8} + \frac{1}{6}S_{21} - J^S + \widetilde{J}^S - \frac{1}{2}S_{19} + \frac{1}{4}S_8 + \frac{41}{32}Y_1 - \frac{95}{128}Y_1^2 + \frac{1889}{13824}Y_1^3 - \frac{5}{32}Y_2 \\
&\quad + \frac{29}{192}Y_1 Y_2 - \frac{55}{1536}Y_2 Y_1^2 - \frac{5}{1728}Y_2^2 + \frac{25}{20736}Y_1 Y_2^2 + \frac{1}{144}Y_3 - \frac{5}{864}Y_1 Y_3 + \frac{25}{20736}Y_1^2 Y_3, \\
p_{66} &= 6Y_1 - \frac{5}{2}(Y_1^2 + Y_2) + \frac{29}{24}Y_1 Y_2 - \frac{5}{144}Y_2^2 + \frac{1}{6}Y_3 - \frac{5}{72}Y_1 Y_3, \\
p_{666} &= \frac{4}{3}Y_3, \\
p_{56} &= -15 + 12Y_1 - \frac{5}{3}Y_1^2 + \frac{41}{48}Y_{2,1} - \frac{35}{96}Y_1 Y_{2,1} - \frac{1}{3456}(Y_{2,1})^2 - \frac{5}{2}Y_2 + \frac{35}{36}Y_1 Y_2 + \frac{5}{432}Y_2 Y_{2,1} \\
&\quad - \frac{43}{576}Y_{3,2} + \frac{109}{3456}Y_1 Y_{3,2} + \frac{1}{82944}Y_{2,1} Y_{3,2} - \frac{5}{10368}Y_2 Y_{3,2} + \frac{1}{432}Y_{4,3} - \frac{5}{5184}Y_1 Y_{4,3}, \\
p_{556} &= 44 + \frac{1}{6}Y_{3,1} - \frac{1}{108}Y_{4,2} + \frac{1}{5184}Y_{5,3}, \\
p_{566} &= 4Y_{2,1} - \frac{2}{3}Y_{3,2} + \frac{1}{36}Y_{4,3}, \\
p_7 &= \frac{1}{2} - \frac{1}{2}S_{11} + \frac{1}{12}Y_1 - \frac{275}{1152}Y_1^2 + \frac{215}{3456}Y_1^3 - \frac{1}{192}Y_{2,1} + \frac{19}{4608}Y_1 Y_{2,1} \\
&\quad - \frac{145}{165888}Y_1^2 Y_{2,1} - \frac{1}{82944}(Y_{2,1})^2 + \frac{11}{1990656}Y_1 (Y_{2,1})^2 - \frac{25}{288}Y_2 + \frac{29}{384}Y_1 Y_2 - \frac{85}{5184}Y_1^2 Y_2 \\
&\quad - \frac{1}{13824}Y_{2,1} Y_2 + \frac{5}{165888}Y_1 Y_2 Y_{2,1} + \frac{1}{4608}Y_{3,2} - \frac{1}{6144}Y_1 Y_{3,2} + \frac{5}{165888}Y_1^2 Y_{3,2} \\
&\quad + \frac{1}{1990656}Y_{2,1} Y_{3,2} - \frac{5}{23887872}Y_1 Y_{2,1} Y_{3,2},
\end{aligned}$$

$$\begin{aligned}
p_8 &= \frac{1}{4} - \frac{1}{2}S_{13} + \frac{7}{48}Y_1 - \frac{121}{576}Y_1^2 + \frac{353}{6912}Y_1^3 - \frac{1}{18}Y_2 + \frac{5}{96}Y_1 Y_2 - \frac{125}{10368}Y_1^2 Y_2, \\
p_9 &= -\frac{9}{8} - \frac{1}{16}(S_{10} + S_{11} + S_{14} - 2S_{15} + S_{16} - 2S_{17}) + \frac{3}{2}Y_1 - \frac{31}{48}Y_1^2 + \frac{13}{144}Y_1^3 + \frac{1}{1152}Y_{2,1} \\
&\quad - \frac{1}{1728}Y_1 Y_{2,1} + \frac{1}{10368}Y_1^2 Y_{2,1} - \frac{1}{12}Y_2 + \frac{1}{16}Y_1 Y_2 - \frac{5}{432}Y_1^2 Y_2, \\
p_{57} &= -\frac{43}{16} + \frac{187}{48}Y_1 + \frac{871}{2304}Y_{2,1} - \frac{1}{6}Y_1 Y_{2,1} + \frac{1}{3456}(Y_{2,1})^2 - \frac{1}{576}Y_{3,1} + \frac{1}{1728}Y_1 Y_{3,1} \\
&\quad - \frac{1}{248832}Y_{2,1} Y_{3,1} - \frac{25}{1728}Y_{3,2} + \frac{17}{2592}Y_1 Y_{3,2} - \frac{1}{82944}Y_{2,1} Y_{3,2} + \frac{1}{27648}Y_{4,2} \\
&\quad - \frac{1}{82944}Y_1 Y_{4,2} + \frac{1}{11943936}Y_{2,1} Y_{4,2}, \\
p_{58} &= -4 + \frac{13}{3}Y_1 + \frac{1}{4}Y_{2,1} - \frac{1}{8}Y_1 Y_{2,1} - \frac{1}{108}Y_{3,2} + \frac{25}{5184}Y_1 Y_{3,2}, \\
p_{59} &= -6 + 2Y_1 + \frac{3}{8}Y_{2,1} - \frac{1}{8}Y_1 Y_{2,1} - \frac{1}{72}Y_{3,2} + \frac{1}{216}Y_1 Y_{3,2}, \\
p_{67} &= 9 + Y_1 - \frac{4}{3}Y_1^2 - \frac{5}{48}Y_{2,1} + \frac{1}{24}Y_1 Y_{2,1} - \frac{1}{3456}(Y_{2,1})^2 - \frac{25}{12}Y_2 + \frac{17}{18}Y_1 Y_2 \\
&\quad - \frac{1}{576}Y_2 Y_{2,1} + \frac{1}{192}Y_{3,2} - \frac{1}{576}Y_1 Y_{3,2} + \frac{1}{82944}Y_{2,1} Y_{3,2}, \\
p_{68} &= 4Y_1 - \frac{4}{3}Y_1^2 - \frac{4}{3}Y_2 + \frac{25}{36}Y_1 Y_2, \\
p_{69} &= 6Y_1 - 2Y_1^2 - 2Y_2 + \frac{2}{3}Y_1 Y_2. \tag{9.3}
\end{aligned}$$

Here a new set of integrals has appeared, S_1 , S_2 , etc., coming from diagrams 4, 5 of Fig. 6. They are defined and calculated in Appendix A. This set of integrals is not completely independent and in Appendix B we give some relationships among them.

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